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# Dynamic Information Disclosure for Deception*

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***Abstract*— We analyze in this paper how a deceptive information provider can shape the shared information in order to control a decision maker's decisions. Data-driven engineering applications, e.g., machine learning and artificial intelligence, build on information. However, this implies that information (and correspondingly information providers) can have influential impact on the decisions made. Notably, the information providers can be deceptive such that they can benefit, while the decision makers suffer, from the strategically shaped information. We formulate (and provide an algorithm to compute) the optimal deceptive shaping policies in the multi-stage disclosure of, *general*, multi-dimensional Gauss-Markov information. To be able to deceive the decision maker, the information provider should anticipate the decision maker's reaction while facing a trade-off between deceiving at the current stage and the ability to deceive in the future stages. We show that optimal shaping policies are linear within the general class of Borel-measurable policies even though the information provider and the decision maker could be seeking to minimize quite different quadratic cost functions.**

## I. INTRODUCTION

In non-cooperative multi-agent environments, agents can leverage the asymmetry of information to impact others' decisions. We seek to address how an agent that has access to some time-variant information, e.g., information provider, can control the decision of another agent by shaping the shared information. The impact of this indirect control attempt can range from negligible to severe depending on the scenarios. For example, the impact can be as severe as fake news during an election [1] or state censorship on media reports to avoid public revolution [2]. Furthermore, time-variance of the information is essential since the information provider faces a trade-off between deceiving now and being able to deceive in the future [3]. Here, we, specifically, focus on deceptive shaping of time-variant information. A detailed review of the literature for time-invariant information can be found in [4].

Recently, in [3], the author has extended the Bayesian persuasion model [5] to dynamic settings, where a principal (information provider) shares shaped information about a stochastic process (over a finite alphabet) to an agent (decision maker) with myopic objectives and a finite set of actions. The principal commits certain shaping policies in order to control the agent's reaction. In [6], the authors have studied the strategic sensor networks, where sensors (information provider) observe Gaussian information and have myopic objectives while the receiver (decision maker) makes a decision according to an affine policy after collecting the sensor outputs. Note that since the information provider has a myopic objective, he/she does not face a trade-off between deceiving now and being able to deceive in the future.

In this paper, we analyze deceptive shaping policies in the disclosure of general multi-dimensional Gauss-Markov processes over a finite horizon, while information provider and the decision maker can seek to minimize different quadratic cost functions. This extends the results in [4] to more general settings, where the innovation in the Gauss-Markov process can be degenerate. Particularly, we provide a unified result for multi-stage disclosure of Gauss-Markov information that can be from independent and identically distributed to completely correlated, i.e., time-invariant, Gaussian processes. We show that the optimal deceptive shaping policies are linear within the general class of Borel-measurable policies.

To this end, we formulate the functional minimization problem faced by the information provider and derive another finite-dimensional minimization problem that bounds the original problem from below by computing certain necessary conditions that the shaping policies should satisfy. This new finite-dimensional problem is indeed a semi-definite programming (SDP) problem. Since computing a closed-form analytical solution for an SDP problem is challenging in general, we have characterized the solutions by exploiting the problem structure. Particularly, since the objective function is linear in the optimization arguments and the constraint set is a convex set, the solution lies at its extreme points. While characterizing the extreme points, since we consider here general Gauss-Markov processes, different from [4], we have exploited the structure of the constraint set further via the Schur complement condition for positive semi-definiteness. After characterization, we show that certain linear shaping policies can attain the minimum of the lower bound, and correspondingly solve the original functional optimization problem within the general class of Borel-measurable policies. We also provide an SDP based algorithm to compute these optimal policies numerically.

We also note that, turning the problem around, deceptive shaping can also be used as a defense measure against an attacker in cyber/cyber-physical systems. Due to the asymmetry of information, how information flows in-between defender and attacker can play a deterministic role in who succeeds or fails. As an example, the attacker seeks to learn the system dynamics based on the system outputs in order to design successful attacks by evading detection mechanisms. However, the defender can deceptively shape the system

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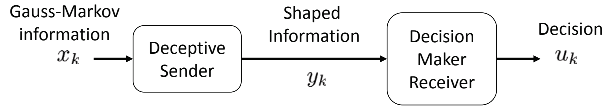


Fig. 1. Deceptive information disclosure model.

outputs to control the attacker's perception for enhanced detection and mitigation of the attack. In that respect, in [7], we have introduced the secure sensor design framework for stochastic control systems that can enhance resiliency prior to detection of any attacks with control objectives. Particularly, sensor outputs are shaped strategically against the possibility of the attacks that could not be detected by the intrusion detection systems.

The paper is organized as follows: We provide the problem formulation and the main results in Section II and III, respectively. We provide numerical examples in Section IV. In Section V, we conclude the paper with several remarks. Two appendices include proofs of some of the technical results.

## II. PROBLEM FORMULATION

Consider a zero-mean discrete-time, exogenous, Gauss-Markov process[1]

$$\{\boldsymbol{x}_k \sim \mathbb{N}(0, \Sigma_k)\} \tag{1}$$

for $k = 1, \dots, n$, where $\Sigma_k \in \mathbb{R}^p$. As seen in Fig. 1, we have two agents: Sender (S) and Receiver (R). At each instant, S shapes the state $\boldsymbol{x}_k \in \mathbb{R}^p$ and sends the shapeed information $\boldsymbol{y}_k \in \mathbb{R}^p$ to R while R makes a decision $\boldsymbol{u}_k \in \mathbb{R}^r$ based on the shapeed information, e.g., estimation of the true state. Particularly, S selects the shaping policies $\eta_1(\cdot), \dots, \eta_n(\cdot)$ from the corresponding policy spaces:

$$\Omega_k := \{\eta : \mathbb{R}^{kp} \to \mathbb{R}^p \mid \eta \text{ is a Borel measurable function}\}.$$

On the other side, R selects the decision policies $\gamma_1, \dots, \gamma_n$ from the corresponding policy spaces:

$$\Gamma_k := \{\gamma : \mathbb{R}^{kp} \to \mathbb{R}^r \mid \gamma \text{ is a Borel measurable function}\}.$$

Note that $\eta_k$'s are Borel measurable functions, therefore $\{\boldsymbol{y}_k\}$ is also a well-defined random process; however, not necessarily Markovian or Gaussian.

[1]**Notations:** For an ordered set of parameters, e.g., $x_1, \cdots, x_n$, we define $x_{[k,l]} := x_k, \cdots, x_l$, where $1 \leq k \leq l \leq n$. $\mathbb{N}(0,.)$ denotes the multivariate Gaussian distribution with zero mean and designated covariance. We denote random variables by bold lower case letters, e.g., $\boldsymbol{x}$. For a vector $x$ and a matrix $A$, $x'$ and $A'$ denote their transposes, and $\|x\|$ denotes the Euclidean ($L^2$) norm of the vector $x$. For a matrix $A$, $\mathrm{tr}\{A\}$ denotes its trace. We denote the identity and zero matrices with the associated dimensions by $I$ and $O$, respectively. For positive semi-definite matrices $A$ and $B$, $A \succeq B$ means that $A - B$ is also a positive semi-definite matrix.

In this non-cooperative environment, the agents have the following quadratic cost functions:

$$J_S(\eta_{[1,n]}; \gamma_{[1,n]}) = \mathbb{E}\left\{\sum_{k=1}^{n} \|Q_{S,k}\boldsymbol{x}_k - R_{S,k}\boldsymbol{u}_k\|^2\right\}, \tag{2}$$

$$J_R(\eta_{[1,n]}; \gamma_{[1,n]}) = \mathbb{E}\left\{\sum_{k=1}^{n} \|Q_{R,k}\boldsymbol{x}_k - R_{R,k}\boldsymbol{u}_k\|^2\right\}, \tag{3}$$

where $Q_{S,k}, Q_{R,k} \in \mathbb{R}^{r\times p}$ and $R_{S,k}, R_{R,k} \in \mathbb{R}^{r\times r}$. We assume that $J_R$ is a strictly convex function of the decision $\boldsymbol{u}_k$, i.e., $R_{R,k}$ is non-singular. As an example, consider the scenario where $\boldsymbol{x}_k = \left[\begin{smallmatrix}\boldsymbol{z}_k \\ \boldsymbol{\theta}_k\end{smallmatrix}\right]$ while $\boldsymbol{z}_k$ and $\boldsymbol{\theta}_k$ are two separate exogenous processes. S and R seek to minimize

$$\mathbb{E}\left\{\sum_{k=1}^{n} \|\boldsymbol{\theta}_k - \boldsymbol{u}_k\|^2\right\} \text{ and } \mathbb{E}\left\{\sum_{k=1}^{n} \|\boldsymbol{z}_k - \boldsymbol{u}_k\|^2\right\},$$

respectively. In other words, R aims to learn the state $\boldsymbol{z}_k$ while S aims to deceive R through the shapeed information so that R perceives the underlying state as the independent process $\boldsymbol{\theta}_k$. Particularly, we say that S is deceptive when S can attain smaller cost while R attains larger cost due to the strategic shaping of the information [8]. Note that the essence of deception is S's ability to conjecture R's reaction so that he/she can select the shaping policies accordingly. Furthermore, the agents interact multiple times. Therefore, while controlling the transparency of the sent information, S faces a trade-off between deceiving R at the current stage and the ability to deceive him/her in the future stages.

Similar to the Bayesian persuasion framework [3], [5], we consider that there is a hierarchy between the agents, i.e., R can know S's shaping policies while making a decision. Furthermore, R's reactions are unobservable and/or noncontratable by S as in [3]. To show the dependence of R's policies on S's policies, we denote them by $\gamma_k(\eta_{[1,k]})(\cdot)$ instead of $\gamma_k(\cdot)$. Then, the interaction between the agents can be modeled as a Stackelberg game [9], where S is the leader that announces his/her policies, and commits to employ them. Formally, a pair of S and R policies: $\left(\eta^*_{[1,n]}, \gamma^*_{[1,n]}\right)$ attains the Stackelberg equilibrium provided that they satisfy (4) and (5).

In the following section, we compute the equilibrium achieving pair of policies.

## III. MAIN RESULT

Let the innovation in the process $\{\boldsymbol{x}_k\}$ be denoted by $\boldsymbol{w}_k := \boldsymbol{x}_{k+1} - \mathbb{E}\{\boldsymbol{x}_{k+1}|\boldsymbol{x}_k\}$ and define

$$A_k := \mathbb{E}\{\boldsymbol{x}_{k+1}\boldsymbol{x}_k'\}\mathbb{E}\{\boldsymbol{x}_k\boldsymbol{x}_k'\}^{\dagger}.$$

Note that the auto-covariance matrix of $\boldsymbol{w}_k$, i.e., $\mathbb{E}\{\boldsymbol{w}_k\boldsymbol{w}_k'\} \succeq O$, can be singular, e.g., $\boldsymbol{w}_k$ can be a degenerate random variable. Furthermore, $A_k \in \mathbb{R}$ can be singular. Importantly, this section generalizes the results in [4] to general Gauss-Markov processes by relaxing the restrictions on the covariance of the innovation, i.e., $\mathbb{E}\{\boldsymbol{w}_k\boldsymbol{w}_k'\}$, and $A_k$. As an

$$\eta^*_{[1,n]} \in \operatorname*{argmin}_{\substack{\eta_k\in\Omega_k,\\ k=1,\dots,n}} \mathbb{E}\left\{\sum_{k=1}^{n} \left\| Q_{S,k}\boldsymbol{x}_k - R_{S,k}\gamma^*_k(\eta_{[1,k]})(\eta_{[1,k]}(\boldsymbol{x}_{[1,k]}))\right\|^2\right\} \tag{4}$$

$$\gamma^*_{[1,n]}(\eta_{[1,n]}) \in \operatorname*{argmin}_{\substack{\gamma_k\in\Gamma_k,\\ k=1,\dots,n}} \mathbb{E}\left\{\sum_{k=1}^{n} \left\| Q_{R,k}\boldsymbol{x}_k - R_{R,k}\gamma_k(\eta_{[1,k]})(\eta_{[1,k]}(\boldsymbol{x}_{[1,k]}))\right\|^2\right\} \tag{5}$$

example, $A_k = O$ and $\mathbb{E}\{\boldsymbol{w}_k\boldsymbol{w}_k'\} = \Sigma_w$ imply independent and identically distributed state while $A_k = I$ and $\mathbb{E}\{\boldsymbol{w}_k\boldsymbol{w}_k'\} = O$ imply not evolving information.

Given the S's shaping policies $\eta_{[1,k]}$ and shapeed information $\boldsymbol{y}_{[1,k]}$, R's reaction $\boldsymbol{u}_k$ is given by

$$\boldsymbol{u}^*_k = (R'_{R,k}R_{R,k})^{-1}R_{R,k}Q_{R,k}\hat{\boldsymbol{x}}_k, \tag{6}$$

where $\hat{\boldsymbol{x}}_k := \mathbb{E}\{\boldsymbol{x}_k|\boldsymbol{y}_{[1,k]}\}$, almost everywhere on $\mathbb{R}^p$. This yields that R's reaction set, i.e., (5), is a singleton. Then, by (6), S faces the following *functional* optimization problem:

$$\min_{\substack{\eta_k\in\Omega_k,\\ k=1,\dots,n}} \mathbb{E}\left\{\sum_{k=1}^{n}\left\|Q_{S,k}\boldsymbol{x}_k - \Delta_k\hat{\boldsymbol{x}}_k\right\|^2\right\}, \tag{7}$$

where $\Delta_k := R_{S,k}(R'_{R,k}R_{R,k})^{-1}R_{R,k}Q_{R,k}$. Due to the law of iterated expectations and since $\hat{\boldsymbol{x}}_k$ is $\sigma$-$\boldsymbol{y}_{[1,k]}$ measurable, after some algebra, (7) can be written as

$$\min_{\substack{\eta_k\in\Omega_k,\\ k=1,\dots,n}} \sum_{k=1}^{n} \operatorname{Tr}\{V_kH_k\} + \Pi_k, \tag{8}$$

where $H_k := \mathbb{E}\{\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k'\}$ while $V_k := \Delta_k'\Delta_k - \Delta_k'Q_{S,k} - Q'_{S,k}\Delta_k$ and $\Pi_k := \operatorname{Tr}\{\Sigma_kQ'_{S,k}Q_{S,k}\}$ do not depend on the optimization arguments.

Note that in (8), S seeks to find $n$ Borel measurable functions, i.e., $\eta_{[1,n]}$, within the corresponding policy spaces $\Omega_1,\dots,\Omega_n$. Instead of variational calculus based approaches, we follow here an alternative approach as step-by-step summarized below:

- Compute necessary conditions on $H_k$
- Formulate another optimization problem bounding the original problem from below
- Exploit the structure of the necessary conditions to characterize the solutions of this new problem for the cases where the innovation in the process can be degenerate and $A_k$ can be singular
- Show that for certain linear S policies, the lower bound could be achieved in (8) and correspondingly those policies can solve the original problem (8).

The following lemma provides an SDP problem that bounds (8) from below by computing semi-definite matrices that satisfy the necessary conditions on $H_k$.

**Lemma 1 [4].** *The following SDP problem bounds* (8) *from below:*

$$\min_{\substack{S_k\in\mathbb{S}^p,\\ k=1,\dots,n}} \sum_{k=1}^{n} \operatorname{Tr}\{V_kS_k\} \tag{9}$$

*subject to $\Sigma_{j+1} \succeq S_{j+1} \succeq A_jS_jA_j'$, for $j = 0,\dots,n-1$, where $S_0 = O$.*

*Proof.* The proof can be found in [4]. □

Based on Lemma 1, if we can find S's policies that lead to a cost in (8), which is the same with the minimum of the lower bound (9), then those policies minimize (8). Therefore, we invoke the following theorem to characterize the minimum of (9).

**Theorem 1.** *For non-singular $V_k \in \mathbb{S}^p$, the solution of* (9), *i.e., $S_1^*,\dots,S_n^*$, satisfies*

$$S^*_k = A_{k-1}S^*_{k-1}A'_{k-1} + U_k\Lambda_k^{1/2}P_k\Lambda_k^{1/2}U_k', \tag{10}$$

*where the diagonal matrix $\Lambda_k \in \mathbb{S}^p$ and the unitary matrix $U_k \in \mathbb{R}^{p\times p}$ are obtained via the eigen decomposition of $\Sigma_k - A_{k-1}S^*_{k-1}A'_{k-1}$, i.e., $\Sigma_k - A_{k-1}S^*_{k-1}A'_{k-1} = U_k\Lambda_kU_k'$, and $P_k \in \mathbb{S}^p$ are certain symmetric idempotent[2] matrices.*

*Proof.* Let $\Psi \subset \times_{k=1}^{n}\mathbb{S}^p$ denote the constraint set in (9). The convexity and compactness of the constraint set $\Psi$ follow from the proof of Theorem 4 in [4], even though, here, $A_k \in \mathbb{R}^{p\times p}$, $k = 1,\dots,n$, can be singular matrices. Correspondingly, the solution, i.e., global minimum, is attained at the extreme points[3] of the constraint set. Therefore, if we can compute the extreme points of the constraint set, we would have characterized the solution of (9).

Let $S_{-k} := \{S_1,\dots,S_{k-1},S_{k+1},\dots,S_n\}$ and consider the sub-constraint set:

$$\Phi_k(S_{-k}) := \{S_k \in \mathbb{S}^p \mid \Sigma_k \succeq S_k,\ S_{k+1} \succeq A_kS_kA_k',\ S_k \succeq A_{k-1}S_{k-1}A'_{k-1}\}. \tag{11}$$

Then, the following lemma shows that if $(S_1^*,\dots,S_n^*)$ is an extreme point of the constraint set, e.g., the solution of (9), then $S_n^*$ should be an extreme point of the sub-constraint set:

$$\Phi_n(S^*_{-n}) = \{S_n \in \mathbb{S}^p \mid \Sigma_n \succeq S_n \succeq A_{n-1}S^*_{n-1}A'_{n-1}\}. \tag{12}$$

**Lemma 2 [4].** *If a tuple $(E_1,\dots,E_n)$ is an extreme point of the constraint set $\Psi$, then $E_k \in \Phi_k(E_{-k})$ is an extreme point of the sub-constraint set $\Phi_k(E_{-k})$.*

*Proof.* The proof can be found in [4]. ■

Therefore, if we compute the extreme points of the sub-constraint set for $S_k$ given $S_{-k}$, we can characterize the

[2] A matrix $P$ is idempotent if $P = P^2$.

[3] An extreme point of a convex set is a point that cannot be written as a convex combination of any other points in the interior of the set.

extreme points of the constraint set $\Psi$, and correspondingly the solution of (9). Note that the sub-constraint set $\Phi_n(S^*_{-n})$ yields

$$\Sigma_n - A_{n-1}S^*_{n-1}A'_{n-1} \succeq S_n - A_{n-1}S^*_{n-1}A'_{n-1} \succeq O. \quad (13)$$

Let $\Sigma_n - A_{n-1}S^*_{n-1}A'_{n-1} \in \mathbb{S}^p$ have the eigen decomposition: $\Sigma_n - A_{n-1}S^*_{n-1}A'_{n-1} = U_n\Lambda_n U'_n$; then we have

$$\Lambda_n \succeq \mathcal{F}_n(S_n) \succeq O, \quad (14)$$

where $\mathcal{F}_n : \mathbb{S}^p \to \mathbb{S}^p$ is a bijective affine transformation defined by

$$\mathcal{F}_n(S) := U'_n(S - A_{n-1}S^*_{n-1}A'_{n-1})U_n, \quad (15)$$

and let $F_n := \mathcal{F}_n(S_n)$.

Note that $\Sigma_n - A_{n-1}S^*_{n-1}A'_{n-1}$ is a positive semi-definite matrix, and therefore, might have zero eigenvalues. Therefore, we partition the diagonal matrix $\Lambda_n$ and $F_n$ as follows:

$$\Lambda_n = \begin{bmatrix} \Lambda_n^{(11)} & O \\ O & O \end{bmatrix} \text{ and } F_n = \begin{bmatrix} F_n^{(11)} & F_n^{(12)} \\ F_n^{(21)} & F_n^{(22)} \end{bmatrix}. \quad (16)$$

We also note that partitions of an arbitrary positive semi-definite matrix $X$ should satisfy the Schur complement condition for positive semi-definiteness [10]:

$$\begin{aligned} & X = \begin{bmatrix} A & B \\ B' & C \end{bmatrix} \succeq O \\ & \Leftrightarrow A \succeq O,\ C - B'A^\dagger B \succeq O,\ (I - AA^\dagger)B = O \\ & \Leftrightarrow C \succeq O,\ A - BC^\dagger B' \succeq O,\ (I - CC^\dagger)B' = O. \quad (17) \end{aligned}$$

Then, (14) yields

$$\Lambda_n - F_n = \begin{bmatrix} \Lambda_n^{(11)} - F_n^{(11)} & -F_n^{(12)} \\ -F_n^{(21)} & -F_n^{(22)} \end{bmatrix} \succeq O, \quad (18)$$

which implies $F_n^{(22)} \preceq O$ by (17). However, $F_n \succeq O$ also implies $F_n^{(22)} \succeq O$ by (17). Therefore, we can conclude that $F_n^{(22)} = O$. Furthermore, the following lemma shows that $F_n^{(12)} = (F_n^{(21)})' = O$, since $F_n^{(22)} = O$ and $\Lambda_n - F_n \succeq O$.

**Lemma 3.** *For a semi-definite matrix $X$ that can be partitioned as $X = \begin{bmatrix} A & B \\ B' & O \end{bmatrix}$, we have*

$$X = \begin{bmatrix} A & B \\ B' & O \end{bmatrix} \succeq O \Leftrightarrow A \succeq O,\ B = O. \quad (19)$$

*Proof.* The proof is provided in Appendix I. ■

By invoking Lemma 3, we obtain

$$\Lambda_n \succeq F_n \succeq O \Leftrightarrow \begin{cases} \Lambda_n^{(11)} \succeq F_n^{(11)} \succeq O_t, \\ F_n^{(12)} = (F_n^{(21)})' = O_{t\times(p-t)}, \\ F_n^{(22)} = O_{p-t}, \end{cases} \quad (20)$$

where $t = \text{rank}(\Sigma_n - A_{n-1}S^*_{n-1}A'_{n-1})$. In particular, in a more explicit form, if $S_n \in \Phi_n(S^*_{-n})$, then

$$\begin{aligned} & \Lambda_n^{(11)} \succeq \begin{bmatrix} I \\ O \end{bmatrix} \mathcal{F}_n(S_n) \begin{bmatrix} I \\ O \end{bmatrix}' \succeq O_t \text{ and} \\ & \mathcal{F}_n(S_n) - \begin{bmatrix} I \\ O \end{bmatrix} \mathcal{F}_n(S_n) \begin{bmatrix} I \\ O \end{bmatrix}' = O_p. \quad (21) \end{aligned}$$

This explicit representation will be helpful while computing the extreme points of the sub-constraint set $\Phi_n(S^*_{-n})$.

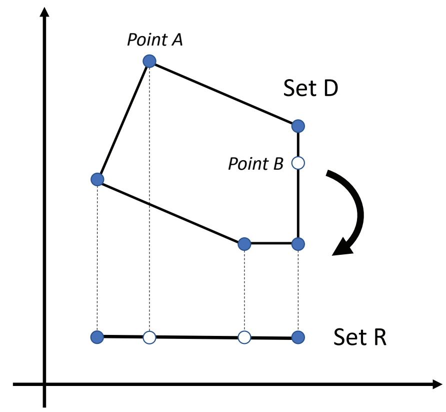


Fig. 2. Over $\mathbb{R}^2$, affine transformation of a compact and convex set D to a compact and convex set R. The filled circles denote the extreme points. Note that under affine, and yet not necessarily bijective, transformation, extreme points are not invariant. As an example, $A$ is an extreme point of Set D while the transformed $A$ is not an extreme point of Set R. Furthermore, $B$ is a non-extreme point of Set D while the transformed $B$ is an extreme point of Set R.

We note that under a bijective affine transformation, the extreme points are mapped to the extreme points of the transformed set [11]. Therefore, if $S_n \in \Phi_n(S^*_{-n})$ is an extreme point of $\Phi_n(S^*_{-n})$, then $F_n = \mathcal{F}_n(S_n)$ is an extreme point of the transformed sub-constraint set $\mathcal{F}_n(\Phi_n(S^*_{-n}))$.

Next, consider the following affine, yet not necessarily bijective, transformation $\mathcal{L}_n : \mathbb{S}^p \to \mathbb{S}^p$:

$$\mathcal{L}_n(F) := (\Lambda_n^\dagger)^{1/2} F (\Lambda_n^\dagger)^{1/2}. \quad (22)$$

Further, let $P := \mathcal{L}_n(\mathcal{F}_n(S_n))$; then (20) implies that the partitions of $P \in \mathbb{S}^p$ should satisfy

$$\begin{aligned} & I_t \succeq P^{(11)} \succeq O_t, \\ & P^{(12)} = (P^{(21)})' = O_{t\times(p-t)}, \\ & P^{(22)} = O_{p-t}. \end{aligned}$$

Therefore, the composition of the transformations $\mathcal{L}_n \circ \mathcal{F}_n$ can map the sub-constraint set $\Phi_n(S^*_{-n})$ to

$$\begin{aligned} \mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S^*_{-n})) = \{ & P \in \mathbb{S}^p \mid I_t \succeq P^{(11)} \succeq O_t, \\ & P^{(12)} = (P^{(21)})' = O_{t\times(p-t)}, \text{ and } P^{(22)} = O_{p-t} \}. \end{aligned}$$

Recall that if $S_n$ is an extreme point of $\Phi_n(S^*_{-n})$, then $F_n = \mathcal{F}_n(S_n)$ is an extreme point of $\mathcal{F}_n(\Phi_n(S^*_{-n}))$ since $\mathcal{F}_n$ is a bijective affine transformation. However, the transformation $\mathcal{L}_n$ is not bijective. And as shown in Fig. 2 via a contradicting example, under affine, and yet not necessarily bijective, transformation, the extreme points may not be preserved. In other words, a transformed extreme point can be a non-extreme point of the transformed set or a transformed non-extreme point can be an extreme point of the transformed set, in general. Therefore, in the following, we examine the transformation in more detail to figure out whether the extreme points would be preserved under $\mathcal{L}_n \circ \mathcal{F}_n$ or not, especially over $\Phi_n(S^*_{-n})$.

Suppose that $F_n$ is an extreme point of $\mathcal{F}_n(\Phi_n(S_{-n}^*))$ while $\mathcal{L}_n(F_n)$ is not an extreme point of $\mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S_{-n}^*))$. This implies that there exist two different $M, N \in \mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S_{-n}^*))$ such that

$$\mathcal{L}_n(F_n) = \nu M + (1-\nu)N, \tag{23}$$

for some $\nu \in (0,1)$. Since $M, N \in \mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S_{-n}^*))$, there exist two different $\bar{M}, \bar{N} \in \mathcal{F}_n(\Phi_n(S_{-n}^*))$ such that $\mathcal{L}_n(\bar{M}) = M$ and $\mathcal{L}_n(\bar{N}) = N$. Then, (23) yields

$$\mathcal{L}_n(F_n) = \nu \mathcal{L}_n(\bar{M}) + (1-\nu)\mathcal{L}_n(\bar{N}). \tag{24}$$

Since $\mathcal{L}_n$ is an affine transformation, (24) leads to

$$\mathcal{L}_n(F_n - \nu\bar{M} - (1-\nu)\bar{N}) = O \tag{25}$$

while $F_n, \bar{M}$, and $\bar{N} \in \mathcal{F}_n(\Phi_n(S_{-n}^*))$, and therefore satisfy (21). Note also that the pseudo inverse of $\Lambda_n$ is given by

$$\Lambda_n^\dagger = \begin{bmatrix} (\Lambda_n^{(11)})^{-1} & O \\ O & O \end{bmatrix} \tag{26}$$

while (21) implies

$$F_n^{(12)} = (F_n^{(21)})' = O,\ F_n^{(22)} = O, \tag{27a}$$

$$\bar{M}^{(12)} = (\bar{M}^{(21)})' = O,\ \bar{M}^{(22)} = O, \tag{27b}$$

$$\bar{N}^{(12)} = (\bar{N}^{(21)})' = O,\ \bar{N}^{(22)} = O. \tag{27c}$$

Therefore, by (22), (25) can also be written as

$$\begin{bmatrix} (\Lambda_n^{(11)})^{-1} & O \\ O & O \end{bmatrix} \begin{bmatrix} F_n^{(11)} - \nu\bar{M}^{(11)} - (1-\nu)\bar{N}^{(11)} & O \\ O & O \end{bmatrix} \begin{bmatrix} (\Lambda_n^{(11)})^{-1} & O \\ O & O \end{bmatrix} = O_p,$$

which implies

$$(\Lambda_n^{(11)})^{-1}(F_n^{(11)} - \nu\bar{M}^{(11)} - (1-\nu)\bar{N}^{(11)})(\Lambda_n^{(11)})^{-1} = O_t,$$

or equivalently,

$$F_n^{(11)} = \nu\bar{M}^{(11)} + (1-\nu)\bar{N}^{(11)}. \tag{28}$$

However, (27) and (28) yield $F_n = \nu\bar{M} + (1-\nu)\bar{N}$, which contradicts with the assumption that $F_n$ is an extreme point of $\mathcal{F}_n(\Phi_n(S_{-n}^*))$ since $\nu \in (0,1)$ and $\bar{M}, \bar{N} \in \mathcal{F}_n(\Phi_n(S_{-n}^*))$. Therefore, we can conclude that if $S_n$ is an extreme point of $\mathcal{F}_n(\Phi_n(S_{-n}^*))$, then $P = \mathcal{L}_n \circ \mathcal{F}_n(S_n)$ is an extreme point of $\mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S_{-n}^*))$.

Furthermore, suppose $P$ is an extreme point of $\mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S_{-n}^*))$. Then, there exists a point $P = \mathcal{L}_n(F)$, i.e., inverse image[4] of $P$, in $\mathcal{F}_n(\Phi_n(S_{-n}^*))$. Assume that $F$ is not an extreme point of $\mathcal{F}_n(\Phi_n(S_{-n}^*))$. This implies that there exist $\bar{M}, \bar{N} \in \mathcal{F}_n(\Phi_n(S_{-n}^*))$ such that $F = \nu\bar{M} + (1-\nu)\bar{N}$, for some $\nu \in (0,1)$. Then, there exist $M, N \in \mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S_{-n}^*))$ such that $M = \mathcal{L}_n(\bar{M})$ and $N = \mathcal{L}_n(\bar{N})$. Note that $M$ and $N$ can be the same even though $\bar{M} \neq \bar{N}$. However, (21) implies that

$$\bar{M}^{(12)} = (\bar{M}^{(21)})' = O \text{ and } \bar{M}^{(22)} = O,$$
$$\bar{N}^{(12)} = (\bar{N}^{(21)})' = O \text{ and } \bar{N}^{(22)} = O.$$

Therefore, $\bar{M}^{(11)} \neq \bar{N}^{(11)}$ since $\bar{M} \neq \bar{N}$; and correspondingly $M \neq N$ due to (22) and (26). However, since $\mathcal{L}_n$ is an affine map, this yields that $P = \nu M + (1-\nu)N$, for some $\nu \in (0,1)$ and $M \neq N$, which contradicts with the assumption that $P$ is an extreme point. Therefore, by contradiction, we can conclude that if $P$ is an extreme point of $\mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S_{-n}^*))$, then there exists $S \in \Phi_n(S_{-n}^*)$ such that $P = \mathcal{L}_n \circ \mathcal{F}_n(S)$ and $S$ is an extreme point of $\Phi_n(S_{-n}^*)$.

[4]The inverse image may not be unique since $\mathcal{L}_n$ is not a bijective map.

Particularly, if we can compute the extreme points of $\bar{\Phi}_n := \mathcal{L}_n \circ \mathcal{F}_n(\Phi_n(S_{-n}^*))$, which can also be written as

$$\bar{\Phi}_n = \left\{ P = \begin{bmatrix} P^{(11)} & P^{(12)} \\ P^{(21)} & P^{(22)} \end{bmatrix} \in \mathbb{S}^p \mid I_t \succeq P^{(11)} \succeq O_t, \right.$$
$$\left. P - P^{(11)} = O_p \right\}, \tag{29}$$

we can compute the extreme points of $\Phi_n(S_{-n}^*)$ and correspondingly we would have characterized the solution of (9). To this end, we invoke the following lemma that characterizes the extreme points of the convex set $\Phi := \{P \in \mathbb{S}^p \mid I \succeq P \succeq O\}$.

**Lemma 4 [4].** *A point $P_e \in \Phi$ is an extreme point if, and only if, $P_e$ is a symmetric idempotent matrix.*

*Proof.* The proof can be found in [4]. ∎

Based on Lemma 4, we can conclude that if $P$ is an extreme point of $\bar{\Phi}$, then $P^{(11)}$ must be a symmetric idempotent matrix, and therefore, $P$ is a symmetric idempotent matrix by (29). This yields that the extreme points of $\Phi_n(S_{-n}^*)$ are given by

$$S_n = A_{n-1}S_{n-1}^*A_{n-1}' + U_n\Lambda_n^{1/2}P_n\Lambda_n^{1/2}U_n', \tag{30}$$

where $P_n$ is a symmetric idempotent matrix. We note that each $S_k^*$ is an arbitrary point in the corresponding sub-constraint set $\Phi_k(S_{-k}^*)$. Therefore, at stage $n-1$, we have the sub-constraint set:

$$\Phi_{n-1}(S_{-(n-1)}^*) = \{S_{n-1} \in \mathbb{S}^p \mid \Sigma_{n-1} \succeq S_{n-1},$$
$$S_n^* \succeq A_{n-1}S_{n-1}A_{n-1}',\ S_{n-1} \succeq A_{n-2}S_{n-2}^*A_{n-2}'\}. \tag{31}$$

If $S_n^*$ is given according to (30), then we have

$$A_{n-1}S_{n-1}A_{n-1} + U_n\Lambda_n^{1/2}P_n\Lambda_n^{1/2}U_n' \succeq A_{n-1}S_{n-1}A_{n-1}$$

since $U_n\Lambda_n^{1/2}P_n\Lambda_n^{1/2}U_n' \succeq O$, and the sub-constraint set $\Phi_{n-1}(S_{-(n-1)}^*)$ can be written as

$$\Phi_{n-1}(S_{-(n-1)}^*) = \{S_{n-1} \in \mathbb{S}^p \mid \Sigma_{n-1} \succeq S_{n-1},$$
$$S_{n-1} \succeq A_{n-2}S_{n-2}^*A_{n-2}'\}. \tag{32}$$

Correspondingly, if $S_n^*$ is an extreme point of $\Phi_n(S_{-n}^*)$, given by (30), then the extreme points of $\Phi_{n-1}(S_{-(n-1)}^*)$ are given by

$$S_{n-1} = A_{n-2}S_{n-2}^*SA_{n-1}' + U_{n-1}\Lambda_{n-1}^{1/2}P_{n-1}\Lambda_{n-1}^{1/2}U_{n-1}$$

where $P_{n-1}$ is a symmetric idempotent matrix. Therefore, following identical steps, we obtain that any extreme point $(S_1^*, \ldots, S_n^*)$ of the constraint set $\Psi$, i.e., the solution of (9), should satisfy (10). □

**Remark.** *If $V_k$'s are singular, there might be other solutions of* (9) *that are non-extreme points of the constraint set $\Psi$. As a trivial example, if all $V_k = O$, any point in the constraint set is a solution of* (9)*. We emphasize that the solutions in the form of* (10) *are essential in our formulation since our main purpose is to compute the best* S *policies, i.e., $\eta_{[1,n]}$, for the original optimization problem* (8) *instead of computing $(S_1^*, \dots, S_n^*)$ for the lower bound* (9)*. And as we will show later in this section, this characterization implies that the lower bound could be achieved via certain* S *strategies. To address non-singularity issues that can arise due to $V_k$'s, we can write the lower bound SDP problem* (9) *as*

$$\min_{(S_1,\dots,S_n)\in\Psi} \lim_{\mu\to 0} \sum_{k=1}^{n} \mathrm{Tr}\{(V_k + \mu I)S_k\}, \tag{33}$$

*or equivalently, as shown in Appendix II,*

$$\lim_{\mu\to 0} \min_{(S_1,\dots,S_n)\in\Psi} \sum_{k=1}^{n} \mathrm{Tr}\{(V_k + \mu I)S_k\}. \tag{34}$$

*Therefore, we can set a $\mu > 0$ such that*[5]

$$\min_{(S_1,\dots,S_n)\in\Psi} \sum_{k=1}^{n} \mathrm{Tr}\{(V_k + \mu I)S_k\} - \min_{(S_1,\dots,S_n)\in\Psi} \sum_{k=1}^{n} \mathrm{Tr}\{V_k S_k\} < \epsilon, \tag{35}$$

*for any $\epsilon > 0$; and the solution of*

$$\min_{(S_1,\dots,S_n)\in\Psi} \sum_{k=1}^{n} \mathrm{Tr}\{(V_k + \mu I)S_k\} \tag{36}$$

*is characterized by* (10) *in Theorem 1.*

Based on Theorem 1, the following theorem shows that for *any* solution of the lower bound (9), e.g., $S_1^*, \dots, S_n^*$, S can select certain linear shaping policy, e.g.,

$$\boldsymbol{y}_k = L_k' \boldsymbol{x}_k, \tag{37}$$

almost everywhere over $\mathbb{R}^p$, such that $H_k = S_k^*$, and correspondingly the linear shaping policies $L_1, \dots, L_n$ minimizes (8).

**Theorem 2.** *Given the solution $S_1^*, \dots, S_n^*$ of the lower bound* (9)*, the corresponding symmetric and idempotent matrices $P_1, \dots, P_n$ could be computed via* (10)*. Let $\Sigma_k - A_{k-1}S_{k-1}^*A_{k-1}'$ and $P_k$ have the eigen decompositions: $\Sigma_k - A_{k-1}S_{k-1}^*A_{k-1}' = U_k\Lambda_k U_k'$ and $P_k = \bar{U}_k\bar{\Lambda}_k\bar{U}_k'$, respectively. Then, linear shaping policy* (37)*, where $L_k$ is given by*

$$L_k = U_k'(\Lambda_k^{1/2})^\dagger \bar{U}_k\bar{\Lambda}_k \tag{38}$$

*yields the hierarchical equilibrium and minimizes* (8) *within general class of policies.*

[5]Note that for all $k = 1, \dots, n$, $S_k \succeq O$, and therefore $\mathrm{Tr}\{S_k\} \geq 0$.

*Proof.* Consider that S has selected the shaping policies according to (37). Correspondingly, $\hat{\boldsymbol{x}}_k$ is given by

$$\begin{aligned} \hat{\boldsymbol{x}}_1 &= \Sigma_1 L_1(L_1'\Sigma_1 L_1)^\dagger L_1'\boldsymbol{x}_1, \\ \hat{\boldsymbol{x}}_k &= A_{k-1}\hat{\boldsymbol{x}}_{k-1} + (\Sigma_k - A_{k-1}H_{k-1}A_{k-1}') \\ &\times L_k(L_k'(\Sigma_k - A_{k-1}H_{k-1}A_{k-1}')L_k)^\dagger L_k'(\boldsymbol{x}_k - A_{k-1}\hat{\boldsymbol{x}}_{k-1}), \end{aligned}$$

for $k \geq 2$ [12]. This yields that $H_k = \mathbb{E}\{\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k'\}$ is given by

$$\begin{aligned} H_1 &= \Sigma_1 L_1(L_1'\Sigma_1 L_1)^\dagger L_1'\Sigma_1, \\ H_k &= A_{k-1}H_{k-1}A_{k-1}' + (\Sigma_k - A_{k-1}H_{k-1}A_{k-1}') \\ &\quad \times L_k(L_k'(\Sigma_k - A_{k-1}H_{k-1}A_{k-1}')L_k)^\dagger L_k' \\ &\quad \times (\Sigma_k - A_{k-1}H_{k-1}A_{k-1}')', \end{aligned} \tag{39}$$

for $k \geq 2$. Then, consider the eigen decomposition of $\Sigma_k - A_{k-1}H_{k-1}A_{k-1}' = U_{k-1}\Lambda_{k-1}U_{k-1}'$. Then, (39) can be written as

$$\begin{aligned} H_1 &= U_1\Lambda_1 U_1 L_1(L_1'U_1\Lambda_1 U_1'L_1)^\dagger L_1'U_1\Lambda_1 U_1, \\ H_k &= A_{k-1}H_{k-1}A_{k-1}' + U_k\Lambda_k U_k' \\ &\qquad \times L_k(L_k'U_k\Lambda_k U_k'L_k)^\dagger L_k'U_k\Lambda_k U_k', \end{aligned} \tag{40}$$

for $k \geq 2$. We point out the resemblance between $S_k$ in (10) and $H_k$ in (39). Particularly, if we let $\mathcal{C}_k := \Lambda_k^{1/2}U_k L_k$, then we can write (40) as

$$\begin{aligned} H_1 &= U_1\Lambda_1^{1/2}\mathcal{C}_1(\mathcal{C}_1'\mathcal{C}_1)^\dagger\mathcal{C}_1'\Lambda_1^{1/2}, \\ H_k &= A_{k-1}H_{k-1}A_{k-1}' + U_k\Lambda_k^{1/2}\mathcal{C}_k(\mathcal{C}_k'\mathcal{C}_k)^\dagger\mathcal{C}_k'\Lambda_k^{1/2}U_k', \end{aligned}$$

for $k \geq 2$.

Note that $C_k := \mathcal{C}_k(\mathcal{C}_k'\mathcal{C}_k)^\dagger\mathcal{C}_k'$ is a symmetric idempotent matrix, which can be partitioned by

$$C_k = \begin{bmatrix} C_k^{(11)} & O_{t\times(p-1)} \\ O_{(p-t)\times t} & O_{p-t} \end{bmatrix}, \tag{41}$$

where $t = \mathrm{rank}(\Sigma_k - A_{k-1}H_{k-1}A_{k-1}')$. Furthermore, recall that any solution (10) of the lower bound should satisfy:

$$S_k^* = A_{k-1}S_{k-1}A_{k-1}' + U_k\Lambda_k^{1/2}\underbrace{\begin{bmatrix} P_k^{(11)} & O_{t\times(p-t)} \\ O_{(p-t)\times t} & O_{p-t} \end{bmatrix}}_{=P_k}\Lambda_k^{1/2}U_k'.$$

If we let $P_k$ have the eigen decomposition: $P_k = \bar{U}_k\bar{\Lambda}_k\bar{U}_k'$ and let S set the shaping policies such that $\mathcal{C}_k = \bar{U}_k\bar{\Lambda}_k$, which implies (38), then we obtain $H_k = S_k^*$ for $k = 1, \dots, n$. Hence, the linear shaping policy (37), where $L_k$ is given by (38), minimizes the original problem (8) within the general class of policies. This completes the proof. □

In Table I, we provide a description of the deceptive shaping algorithm.

## IV. Illustrative Examples

As numerical illustrations, we consider specific dynamic information disclosure scenarios where $\boldsymbol{x}_k = \begin{bmatrix} \boldsymbol{z}_k \\ \boldsymbol{\theta}_k \end{bmatrix} \in \mathbb{R}^2$ while $\boldsymbol{z}_k \in \mathbb{R}$ and $\boldsymbol{\theta}_k \in \mathbb{R}$ are two separate stationary exogenous processes. S and R seek to minimize

$$\mathbb{E}\left\{\sum_{k=1}^{n} \|\boldsymbol{z}_k + \boldsymbol{\theta}_k - \boldsymbol{u}_k\|^2\right\} \text{ and } \mathbb{E}\left\{\sum_{k=1}^{n} \|\boldsymbol{z}_k - \boldsymbol{u}_k\|^2\right\},$$

TABLE I

A DESCRIPTION TO COMPUTE OPTIMAL DECEPTIVE SHAPING POLICIES.

**Algorithm:** Deceptive shaping

**SDP Problem:**

*Compute* $V_k$, $\forall k$.

*Solve the SDP problem* (9) *through a numerical toolbox and obtain the solution* $S_k^*$, $\forall k$.

*Set* $S_0^* = O$.

**Equilibrium achieving policies:**

*Compute the corresponding idempotent matrices* $P_k, \forall k$, *by using* $S_k^*$, $\forall k$, *and* (10).

*Compute the eigen decompositions:*

$\Sigma_k - A_{k-1} S_{k-1}^* A_{k-1} = U_k \Lambda_k U_k'$.

*Compute the eigen decompositions:* $P_k = \bar{U}_k \bar{\Lambda}_k \bar{U}_k'$.

*Compute* $L_k$, $\forall k$, *by using* $\bar{U}_k, \bar{\Lambda}_k, U_k, \Lambda_k$, *and* (38).

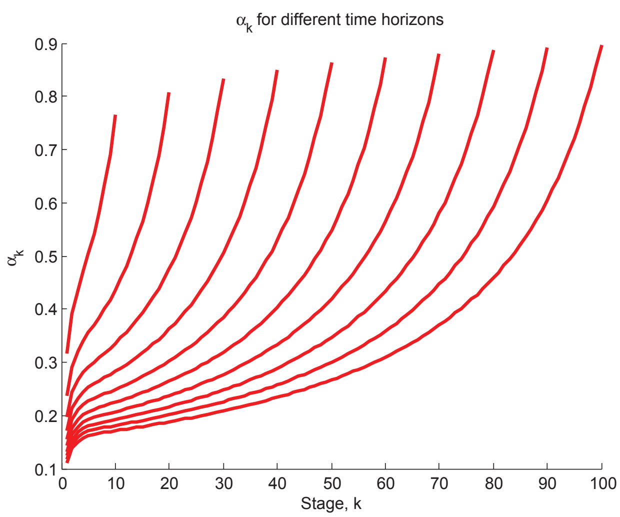


Fig. 3. Scenario 1: the process $\{\boldsymbol{z}_k\}$ evolves according to (42) while $\{\boldsymbol{\theta}_k\}$ is time invariant. The weight of $\boldsymbol{\theta}_k$ in the disclosed information $\boldsymbol{z}_k + \alpha_k \boldsymbol{\theta}_k$ increases over time.

respectively. Note that these scenarios generalize the information disclosure setup in [13] to dynamic settings. We consider two different scenarious:

In Scenario 1, the stationary process $\{\boldsymbol{z}_k \sim \mathbb{N}(0,1)\}$ evolves according to

$$\boldsymbol{z}_{k+1} = \frac{1}{2}\boldsymbol{z}_k + \boldsymbol{w}_k, \tag{42}$$

where $\{\boldsymbol{w}_k \sim \mathbb{N}(0, 3/4)\}$ is a white Gaussian process that is independent of all the other parameters, and the process $\{\boldsymbol{\theta}_k = \boldsymbol{\theta} \sim \mathbb{N}(0,1)\}$ is time-invariant. Particularly, while R seeks to learn a dynamic process, S wants R to perceive the process as its time-invariant shift. Even though the disclosed information is 2-dimensional, $\boldsymbol{y}_k = L_k \boldsymbol{x}_k \in \mathbb{R}^2$, we have observed that all optimal $L_k$'s turn out to have rank 1. Correspondingly, the disclosed information can be written as $\boldsymbol{z}_k + \alpha_k \boldsymbol{\theta}_k$, where $\alpha_k \in \mathbb{R}$ is a certain constant. In Fig. 3, we plot $\alpha_k$ for different time horizons, e.g., $10, 20, \ldots, 100$, which shows that the weight of $\boldsymbol{\theta}_k$ increases in time. Since R can learn the static $\boldsymbol{\theta}_k$ and can cancel out it in the disclosed information to some extent, the disclosed information becomes less and less informative in terms of $\boldsymbol{z}_k$ compared to previously disclosed information.

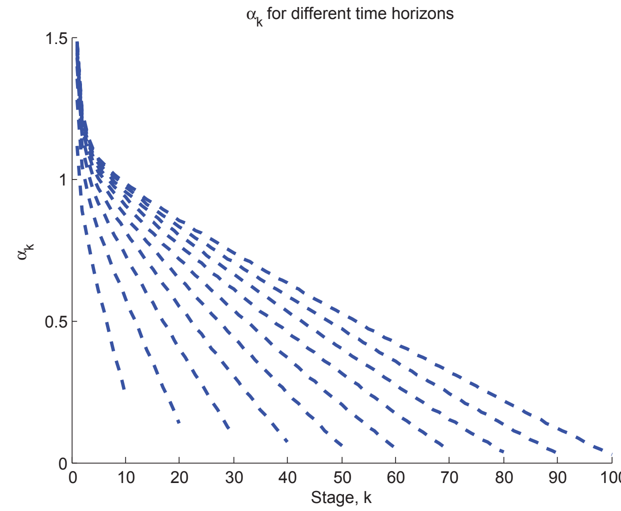


Fig. 4. Scenario 2: the process $\{\boldsymbol{z}_k\}$ is time invariant while $\{\boldsymbol{\theta}_k\}$ evolves according to (43). The weight of $\boldsymbol{\theta}_k$ in the disclosed information $\boldsymbol{z}_k + \alpha_k \boldsymbol{\theta}_k$ decreases over time.

In Scenario 2, we consider the reverse of Scenario 1, i.e., now, the process $\{\boldsymbol{z}_k = \boldsymbol{z} \sim \mathbb{N}(0,1)\}$ is time invariant while the stationary process $\{\boldsymbol{\theta}_k \sim \mathbb{N}(0,1)\}$ evolves according to

$$\boldsymbol{\theta}_{k+1} = \frac{1}{2}\boldsymbol{\theta}_k + \boldsymbol{v}_k, \tag{43}$$

where $\{\boldsymbol{v}_k \sim \mathbb{N}(0, 3/4)\}$ is a white Gaussian process that is independent of all the other parameters. In other words, while R seeks to learn a static parameter, S wants R to perceive the process as its time-variant shift. We have again observed that all optimal $L_k$'s turn out to have rank 1. In Fig. 4, we plot $\alpha_k$ in the disclosed information $\boldsymbol{z}_k + \alpha_k \boldsymbol{\theta}_k$ for different time horizons: $n = 10, 20, \ldots, 100$, which shows that the weight of $\boldsymbol{\theta}_k$ decreases in time. Note that R can learn the static $\boldsymbol{z}_k$ to some extent. Since S aims $\boldsymbol{u}_k$ to be close to $\boldsymbol{\theta}_k$, the disclosed information $\boldsymbol{z}_k + \alpha_k \boldsymbol{\theta}_k$ should have more impact on R's perception, i.e., $\boldsymbol{u}_k$, and correspondingly should be relatively more informative in terms of $\boldsymbol{z}_k$ compared to previously disclosed information.

## V. CONCLUSION

In this paper, we have analyzed how a deceptive information provider can shape a general (multi-dimensional) Gauss-Markov process in order to control a decision maker's decisions over finite horizon. The information provider should control the transparency of the shared information to deceive the decision maker now and to be able to deceive in the future. By considering general Gauss-Markov processes, we have provided a unified result for various type of information, from an independently and identically distributed to time-invariant Gaussian processes. We have shown that the optimal deceptive shaping policies are linear within the general class of Borel-measurable policies even though the information provider and the decision maker can seek to minimize quite different quadratic cost functions. We have also provided an SDP based algorithm to compute the optimal policies numerically. Some future directions of research on this topic include analysis of deceptive shaping policies over an infinite decision-making horizon and dynamic disclosure of continuous-time Gauss-Markov information.

## Appendix I
## Proof of Lemma 2

Let us first consider a 2-dimensional matrix $M_2 := \left[\begin{smallmatrix} a & b \\ b & 0 \end{smallmatrix}\right] \succeq O$. Then, by (17), we have[6]

$$\left[\begin{smallmatrix} a & b \\ b & 0 \end{smallmatrix}\right] \succeq O \Leftrightarrow a \geq 0, \text{ and} \begin{cases} -b^2 a \geq 0 & \text{if } a > 0 \\ b = 0 & \text{if } a = 0. \end{cases} \quad (44)$$

However, if $a > 0$, then $-b^2 a \geq 0$ implies $b = 0$; and if $a = 0$, then $b = 0$. Therefore, (44) is equivalent to

$$\left[\begin{smallmatrix} a & b \\ b & 0 \end{smallmatrix}\right] \succeq O \Leftrightarrow a \geq 0, \; b = 0. \quad (45)$$

Alternatively, determinant is non-negative if, and only if, $b = 0$ and non-negative determinant is a necessary condition for positive semi-definiteness.

Next, consider a 3-dimensional matrix

$$M_3 := \left[\begin{smallmatrix} a & b & c \\ b & d & e \\ c & e & 0 \end{smallmatrix}\right] \succeq O.$$

Then, (17) implies that the right bottom sub-matrix $\left[\begin{smallmatrix} d & e \\ e & 0 \end{smallmatrix}\right] \succeq O$. Therefore, by (45), we can conclude that $e = 0$, which yields

$$M_3 = \left[\begin{smallmatrix} a & b & c \\ b & d & 0 \\ c & 0 & 0 \end{smallmatrix}\right].$$

Through the similarity transformation of $M_3$ with a certain permutation matrix $P$, we obtain $PM_3P'$ as

$$\bar{M}_3 := \left[\begin{smallmatrix} & 1 & \\ 1 & & \\ & & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} a & b & c \\ b & d & e \\ c & e & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} & 1 & \\ 1 & & \\ & & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} d & b & 0 \\ b & a & c \\ 0 & c & 0 \end{smallmatrix}\right].$$

Note that positive semi-definiteness is preserved under similarity transformation [14]. However, by (17), $\bar{M}_3 \succeq O$ implies that the right bottom sub-matrix $\left[\begin{smallmatrix} a & c \\ c & 0 \end{smallmatrix}\right] \succeq O$, and (45) yields $c = 0$. Therefore, we have

$$\left[\begin{smallmatrix} a & b & c \\ b & d & e \\ c & e & 0 \end{smallmatrix}\right] \succeq O \Leftrightarrow \left[\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right] \succeq O \text{ and } c, e = 0.$$

By induction, we can conclude that, for $A \in \mathbb{R}^{p \times p}$ and $\underline{b} \in \mathbb{R}^p$, we obtain

$$\left[\begin{smallmatrix} A & \underline{b} \\ \underline{b}' & 0 \end{smallmatrix}\right] \succeq O \Leftrightarrow A \succeq O \text{ and } \underline{b} = \underline{0}. \quad (46)$$

Based on (46) and through the similarity transformations with certain permutation matrices, we can also obtain (19). This completes the proof.

[6]For a scalar $a \in \mathbb{R}$, if $a = 0$, $a^\dagger = 0$, else $a^\dagger = 1/a$.

## Appendix II
## Equivalence of (33) and (34):

Consider the function $J : [0,1] \times \Psi \to \mathbb{R}$, given by

$$J(\mu, S) := \sum_{k=1}^{n} \mathrm{Tr}\{(V_k + \mu I)S_k\}, \quad (47)$$

where $S := (S_1, \ldots, S_n)$. Then, the function is uniformly continuous on the compact set $[0,1] \times \Psi$, which implies that, for any $(\mu_o, S_o), (\mu, S) \in [0,1] \times \Psi$, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$J(\mu_o, S_o) < J(\mu, S) + \epsilon \quad (48)$$

if[7] $\|S - S_o\| < \delta$ and $|\mu - \mu_o| < \delta$. Furthermore, consider the function $G : [0,1] \to \mathbb{R}$, given by

$$G(\mu) := \min_{(S_1,\ldots,S_n) \in \Psi} \sum_{k=1}^{n} \mathrm{Tr}\{(V_k + \mu I)S_k\}. \quad (49)$$

Then, we have $G(\mu_o) \leq J(\mu_o, S_o)$ and there exists $S \in \Psi$ such that $J(\mu, S) = G(\mu)$. Therefore, by (48), we obtain that given $\epsilon > 0$, there exists $\delta > 0$ such that $G(\mu_o) < G(\mu) + \epsilon$ if $|\mu - \mu_o| < \delta$. Symmetry of the arguments yields that $|G(\mu_o) - G(\mu)| < \epsilon$ when $|\mu - \mu_o| < \delta$; hence $G$ is a continuous function of $\mu$ and the problems (33) and (34) are equivalent.

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[7]We can consider any norm for $S$ since norms are equivalent up to constant factors on finite dimensional vector spaces.