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# Deceptive Multi-dimensional Information Disclosure over a Gaussian Channel*

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*Abstract*— **We analyze deceptive multi-dimensional information disclosure over a channel between an information provider and a decision maker. The information provider has access to noisy versions of an underlying information. Different from the classical communication models, the provider has a different (hidden) objective while he/she must still honestly and transparently provide the information for his/her reputation. However, how well the provider has access to the information is private to him/her. We address how he/she can exploit this asymmetry according to his/her deceptive goal by modeling the interaction as a Stackelberg game, where the information provider is the leader. With quadratic objective functions, multi-variate Gaussian information and additive Gaussian noise channel, we analytically formulate the optimal linear deception strategy and the corresponding optimal decision strategy.**

## I. INTRODUCTION

In the era of information, new enterprises have specialized to collect huge amount of data related to certain phenomena that is essential for the decision mechanisms, to process the data via enhanced computational capabilities according to the decision makers' needs, and to provide the processed information with certain quality guarantees. As an example, in [1], the market participants can acquire the information about the estimated real-time price of the electricity in order to compute their acceptability set in the introduced cash-settled options market. However, the essence of the information in the decisions made arises the possibility of manipulation. Can an information provider deceive the decision maker via the provided information so that the decision moves in the direction the information provider has desired? Particularly, can an information provider control the decision maker's perception about the desired information according to certain hidden goals? For example, in the scenario of [1], as mentioned above, can the estimated price provider control the acceptability set of a market participant maliciously for the benefit of the other? Furthermore, in such a business transaction, trustworthiness is essential for the reputation of the information provider. Then, under the constraint of reputation, i.e., to provide the committed quality guarantee in the provided information, could the information provider still deceive the decision maker? To this end, we seek to address, in this paper, these, possible, issues from a deceptive information provider's perspective, e.g., how the information provider can deceive the decision maker, so that this study can lead to new studies that provide secure, i.e., not deceptive, information transmission for unmanipulated decisions.

Strategic information transmission between a sender and a receiver with misaligned objectives was originally introduced in [2] and studied in various applications including advertising to expert advise sharing problems [3]–[5]. In [2], the sender's objective has an additive bias term, commonly known by the sender and the receiver, while the receiver's objective is independent of the bias term. In particular, the sender seeks to shift the receiver's perception about the underlying information by the amount of the bias. Under Nash equilibrium [6], the authors have shown that only a quantization-based mapping of the scalar information drawn from a bounded distribution leads to an equilibrium. In [7], the authors have extended the strategic information transmission to multi-dimensional settings for quadratic loss functions by relaxing the boundedness assumption about the underlying distribution. They have also studied the multi-dimensional information disclosure over same dimensional additive noise channel and obtained conditions under which an informative affine equilibrium exists.

Recently, the strategic information transmission in a hierarchical setting, where the information provider is truthful about the content of the provided information, has attracted substantial interest in control theory, information theory, and economics [8]–[14]. In [9], the authors study strategic sensor networks with perfect channels for Markov-Gaussian processes and with myopic quadratic objective functions, i.e., the players construct strategies just for the current stage irrespective of the length of the horizon, by restricting the receiver strategies to affine functions. Reference [8] addresses the optimality of linear sender strategies within the general class of policies for myopic quadratic objectives. In [7] and [10], the authors show that for scalar parameters, quadratic loss functions, and a commonly known bias parameter, the hierarchical game formulation can be converted into a team problem. Reference [11] shows that linear sender strategies for scalar Gaussian information can achieve the equilibrium within the general class of policies even with additive Gaussian noise channels. In [12], the author demonstrates the optimality of linear sender strategies also for the multi-variate Gaussian information, and with quadratic loss functions. In [13], the authors show the optimality of linear strategies also for Markov-Gaussian processes with finite horizon quadratic loss functions both in the communication and control settings. Recently, based on the strategic information transmission in a hierarchical structure, in [14], the authors have introduced secure sensor

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design, which is a passive security mechanism for cyber-physical systems against the advanced persistent threats that are difficult to detect, i.e., the sensor outputs are designed strategically by anticipating the possibility of undetected threats.

In this paper, we analyze multi-dimensional information disclosure between a deceptive information provider and a decision maker over an additive Gaussian noise channel with a power constraint. Due to the deceptive objective of the information provider, this scheme differs from the classical communication problems over a channel. We consider the underlying information drawn from a multi-variate Gaussian distribution and the information provider has access to (jointly Gaussian) noisy versions of the underlying information. In addition to the power constraint on the sent signal, the information provider also has a reputation constraint such that he/she must be transparent about the content of the sent signal and honest about the quality of the disclosed information, i.e., about the relation of the sent signal and the underlying information. Therefore, the interaction between the information provider and the decision maker can be modeled as a Stackelberg game, where the information provider is the leader of the game by announcing his/her strategies beforehand, i.e., by being transparent about the content of the sent signal. Here, for linear information provider strategies, we analytically formulate the optimal deception strategies and the corresponding optimal decision strategies. We emphasize that this study differs from [12] and [9] due to the additive noise channel, differs from [11] due to the multi-dimensional information, and differs from [7] due to the general quadratic loss functions and privately, i.e., not commonly known, bias parameter such that the scheme does not lead to a team problem in the hierarchical setting and due to analytical formulation of the equilibrium achieving strategies under linearity constraint rather than characterization of the conditions where an informative affine equilibrium exists.

The main contributions of this paper are as follows:

- We study deceptive multi-dimensional information disclosure over a Gaussian channel in a Stackelberg game setting, where the information provider is the leader.
- We model the objective of the deceptive information provider as a convex combination of the decision maker's objective and a deceptive objective such that depending on the weights in the convex combination the degree of misalignment between the objectives varies.
- We formulate the optimal linear deception strategies and the corresponding decision strategies in closed form.

The paper is organized as follows: In Section II, we formulate the deceptive multi-dimensional information disclosure problem and the corresponding game between the information provider and the decision maker. In Section III, we derive the optimal linear decision and the optimal decision strategies analytically. We provide illustrative examples in Section IV. We conclude the paper in Section V with several remarks. Appendices provide proofs for technical results.

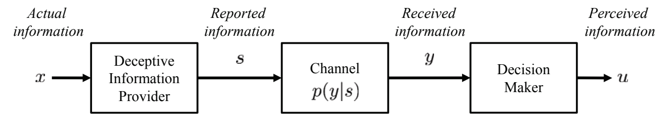


Fig. 1. Deceptive information disclosure.

## II. PROBLEM FORMULATION

Consider two agents: a deceptive information provider and a decision maker, as seen in Fig. 1. The deceptive information provider has access to noisy versions of certain multi-dimensional information[1] $x \in \mathbb{R}^n$, $n \geq 1$, in which the decision maker is interested. The information provider reports $s \in \mathbb{R}$ and its relation with $x$. Furthermore, this reported information can be received by the decision maker after passing through a channel, where $p(y|s)$ represents the probability of receiving $y \in \mathbb{R}$ given that $s$ has been reported. Based on the received information $y$ and its relation with the actual information $x$, the decision maker constructs an estimate of $x$, which is denoted by $\hat{x} \in \mathbb{R}^n$. In particular, $\hat{x}$ is the decision maker's perception of $x$ and different from the classical communication setting, here the deceptive information provider seeks to control the decision maker's perception according to his/her own, possibly malicious, objective.

In this paper, we specifically consider that the multi-dimensional information is a realization of a multi-variate Gaussian random vector $\boldsymbol{x} \sim \mathbb{N}(0, \Sigma_x)$. The information provider observes noisy versions of the information: $z_1, \ldots, z_m \in \mathbb{R}^n$, $m \geq 1$, that are independent realizations of the random vector $\boldsymbol{z} = Ax + \boldsymbol{v}$, where[2] $A \in \mathbb{R}^{n \times n}$, $\boldsymbol{v} \sim \mathbb{N}(0, \Sigma_v)$, and $\boldsymbol{v}$ is independent of $\boldsymbol{x}$. After observing $z := z_1, \ldots, z_m$, the information provider can construct $s = \eta(z)$ such that $\eta(\cdot)$ is a linear function from $\mathbb{R}^{mn}$ to $\mathbb{R}$. Then, the reported information $s$ passes through an additive Gaussian noise channel, i.e., $\boldsymbol{y} = \boldsymbol{s} + \boldsymbol{w}$, where $\boldsymbol{w} \sim \mathbb{N}(0, \sigma_w^2)$ and $\boldsymbol{w}$ is independent of both $\boldsymbol{x}$ and $\boldsymbol{v}$. After receiving $y$, the decision maker constructs $\hat{x} = \gamma(y)$, where $\gamma(\cdot)$ is a Borel measurable function from $\mathbb{R}$ to $\mathbb{R}^n$. In this stochastic setting, we assume that both agents commonly know that

- statistics of the information $\boldsymbol{x}$ and the channel noise $\boldsymbol{w}$ are known by both agents;
- the observation of the information provider is an affine function of $x$ and the reported information $s$ is affine in $z$;

[1]**Notations:** $\mathbb{N}(0, .)$ denotes Gaussian distribution with zero mean and designated (positive-definite) covariance matrix or variance depending on whether it is multi-variate or not. We denote random variables by bold lower case letters and their realizations by the same lower case letters, e.g., $\boldsymbol{x}$ and $x$. For a vector $x$ and a matrix $A$, $x'$ and $A'$ denote their transposes, respectively, and $\|x\|$ denotes the Euclidean ($L^2$) norm of the vector $x$. For a matrix $A$, $\mathrm{tr}\{A\}$ denotes its trace. We denote the identity and zero matrices with the associated dimensions by $I$ and $O$, respectively. For positive semi-definite matrices $A$ and $B$, $A \succeq B$ means that $A - B$ is also a positive semi-definite matrix. We note that all the random parameters have zero mean; however, the derivations can straight-forwardly be extended to non-zero mean case.

[2]Note that we do not make any assumption about the rank of $A$.

- the content of $z$, i.e., how $z$ and $x$ are related, is known only by the information provider;
- the information provider is committed to be truthful about the content of the reported information. In particular, since the reported information $\boldsymbol{s}$ and the actual information $\boldsymbol{x}$ are jointly Gaussian, the information provider discloses truthfully both first and second order statistics of this joint distribution;
- the decision maker is not aware of, or not interested in, whether the information provider has a hidden goal or not since the information provider does not lie about the content of the reported information while the information provider knows the decision maker's objective.

In a classical communication scenario, the objectives of the agents would be aligned and the information provider would choose $\eta(\cdot)$ to mitigate the impact of the channel. However, here, the deceptive information provider has a different objective. Particularly, the information provider wants the decision maker to perceive $x$ as his/her private information $\theta \in \mathbb{R}^n$, which is a realization of $\boldsymbol{\theta} \sim \mathbb{N}(0, \Sigma_\theta)$. Therefore, the deceptive information provider constructs the reported information via $s = \hat{\eta}(z, \theta)$, where $\hat{\eta}(\cdot)$ is a linear function from $\mathbb{R}^{(m+1)n}$ to $\mathbb{R}$. We consider that $\boldsymbol{\theta}$ is jointly Gaussian with $\boldsymbol{x}$ and can be *dependent* or *independent* of $\boldsymbol{x}$, and yet is independent of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Let $\Omega$ and $\Gamma$ denote the set of all linear functions from $\mathbb{R}^{(m+1)n}$ to $\mathbb{R}$ and the set of all Borel measurable functions from $\mathbb{R}$ to $\mathbb{R}^n$, respectively. Then, the deceptive information provider seeks to control the decision maker's perception of $x$ such that

$$\min_{\hat{\eta} \in \Omega} \lambda \mathbb{E}\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_R^2 + (1-\lambda)\mathbb{E}\|\boldsymbol{\theta} - \hat{\boldsymbol{x}}\|_Q^2, \tag{1}$$

subject to power constraint:

$$\mathbb{E}\{\boldsymbol{s}^2\} \le P, \tag{2}$$

where $\lambda \in [0,1]$ and[3] $Q, R \in \mathbb{S}^n$ are positive-definite matrices. On the other hand, the decision maker seeks to estimate the actual information $x$ such that

$$\min_{\gamma \in \Gamma} \mathbb{E}\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_R^2. \tag{3}$$

Note that since $R$ is positive-definite, there exists a unique solution of (3). Furthermore, $\lambda$ determines how much the objectives of the agents are aligned. As an example, $\lambda = 1$ implies a classical communication scenario, where the agents have the same objective.

We note that by knowing the first and second order statistics of the joint distribution of $\boldsymbol{x}$ and $\boldsymbol{s}$, the decision maker knows the content of the received information, i.e., the relation between $y$ and $x$, since they are jointly Gaussian. This implies that there is a hierarchy between the agents. Therefore, the interaction between the non-cooperative agents can be analyzed as a Stackelberg game [6], where the information provider takes action by constructing $s = \hat{\eta}(z, \theta)$ and the decision maker takes action by constructing[4] $\hat{x} = \gamma(y; \hat{\eta})$. Furthermore, the information provider is the leader of the game, by announcing his/her strategy beforehand; and the decision maker is the follower. Note that we can also view this as the information provider choosing a strategy $\hat{\eta}$ from the associated strategy space $\Omega$ and the decision maker choosing a strategy $\gamma$ from the strategy space $\Gamma$, and corresponding to the objective functions (1) and (3), there exist certain cost functions depending on the agents' strategies $\hat{\eta}$ and $\gamma$: $J_L(\hat{\eta}, \gamma)$ and $J_F(\hat{\eta}, \gamma)$ while each strategy implicitly depends on the other. This would be the normal (strategic) form description of the underlying game [6]. Therefore, the pair of strategies $[\hat{\eta}^*, \gamma^*]$ attains the Stackelberg equilibrium provided that

$$\hat{\eta}^* = \operatorname*{argmin}_{\hat{\eta} \in \Omega} J_L(\hat{\eta}, \gamma^*(\cdot; \hat{\eta})) \text{ s.t. } \mathbb{E}\{\hat{\eta}(\boldsymbol{z}, \boldsymbol{\theta})^2\} \le P, \tag{4a}$$

$$\gamma^*(\cdot, \hat{\eta}) = \operatorname*{argmin}_{\gamma \in \Gamma} J_F(\hat{\eta}, \gamma(\cdot; \hat{\eta})). \tag{4b}$$

## III. MAIN RESULT

In this section, we formulate the optimal deception strategies for the information provider analytically in closed form. To this end, we first aim to compute the decision maker's perception of $x$ given the received information $y$. By (3), and since $R \succeq O$, the best reaction of the decision maker is given by $\hat{x}^* = \mathbb{E}\{\boldsymbol{x}|\boldsymbol{y} = y\}$ and correspondingly, we have $\hat{\boldsymbol{x}}^* = \mathbb{E}\{\boldsymbol{x}|\boldsymbol{y}\}$ almost everywhere on $\mathbb{R}^n$. Then, the optimization problem faced by the information provider is given by

$$\min_{\hat{\eta} \in \Omega} \lambda \mathbb{E}\|\boldsymbol{x} - \hat{\boldsymbol{x}}^*\|_R^2 + (1-\lambda)\mathbb{E}\|\boldsymbol{\theta} - \hat{\boldsymbol{x}}^*\|_Q^2 \tag{5}$$

subject to (2).

We note that $\boldsymbol{s} = \hat{\eta}(\boldsymbol{z}, \boldsymbol{\theta})$ can be written as

$$\boldsymbol{s} = c_z' \boldsymbol{z} + c_\theta' \boldsymbol{\theta}, \tag{6}$$

where $c_z \in \mathbb{R}^{mn}$ and $c_\theta \in \mathbb{R}^n$, almost everywhere on $\mathbb{R}$. Then, (5) can be written as

$$\begin{aligned}\min_{c_z \in \mathbb{R}^{mn}, c_\theta \in \mathbb{R}^n} & \lambda \mathbb{E}\|\boldsymbol{x} - \mathbb{E}\{\boldsymbol{x}|c_z'\boldsymbol{z} + c_\theta'\boldsymbol{\theta} + \boldsymbol{w}\}\|_R^2 \\ & + (1-\lambda)\mathbb{E}\|\boldsymbol{\theta} - \mathbb{E}\{\boldsymbol{x}|c_z'\boldsymbol{z} + c_\theta'\boldsymbol{\theta} + \boldsymbol{w}\}\|_Q^2 \end{aligned} \tag{7}$$

subject to

$$\mathbb{E}\{(c_z'\boldsymbol{z} + c_\theta'\boldsymbol{\theta})^2\} \le P \tag{8}$$

and

$$\mathbb{E}\{\boldsymbol{x}|c_z'\boldsymbol{z} + c_\theta'\boldsymbol{\theta} + \boldsymbol{w}\} = \frac{(\mathbb{E}\{\boldsymbol{x}\boldsymbol{z}'\}c_z + \mathbb{E}\{\boldsymbol{x}\boldsymbol{\theta}'\}c_\theta)(c_z'\boldsymbol{z} + c_\theta'\boldsymbol{\theta})}{\mathbb{E}\{(c_z'\boldsymbol{z} + c_\theta'\boldsymbol{\theta})^2\} + \sigma_w^2}.$$

We emphasize that even though (7) is a finite dimensional optimization problem, it is highly nonlinear and non-convex with a quadratic constraint unless $\lambda = 1$. Therefore, numerical computation of the solution requires exhaustive search over $\mathbb{R}^{(m+1)n}$ and it may not lead to the global minimum. However, in the following, we provide a closed form solution for the problem by formulating an optimization

[3] $\mathbb{S}^n$ denotes the set of symmetric $n \times n$ matrices.

[4] We denote the decision maker's strategy by $\hat{x} = \gamma(y; \hat{\eta})$ instead of $\hat{x} = \gamma(y)$ in order to show the dependence on $\hat{\eta}$ explicitly.

problem bounding the original problem from below, solving that new problem in closed form, and then showing that the minimum can be achieved in the original problem via certain vectors $c_z^*, c_\theta^*$, which implies that they are also the solution of the original optimization problem (7). The following theorem provides the analytically computed, equilibrium achieving, strategies.

**Theorem 1.** *Consider multi-dimensional information disclosure over additive Gaussian noise channel. Let* $\boldsymbol{u} := \left[\begin{smallmatrix}\boldsymbol{x}\\ \boldsymbol{\theta}\end{smallmatrix}\right]$, $\boldsymbol{t} := \left[\begin{smallmatrix}\boldsymbol{z}\\ \boldsymbol{\theta}\end{smallmatrix}\right]$ *and* $\Sigma_u := \mathbb{E}\{\boldsymbol{u}\boldsymbol{u}'\}$, $\Sigma_t := \mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}$. *Furthermore, let*

$$W := \frac{P}{P+\sigma_w^2}\Sigma_t^{-1/2}\bar{A}\Sigma_u V \Sigma_u \bar{A}'\Sigma_t^{-1/2}, \tag{9}$$

*where*[5]

$$V := \begin{bmatrix} -\lambda R + (1-\lambda)Q & -(1-\lambda)Q \\ -(1-\lambda)Q & O \end{bmatrix} \text{ and } \bar{A} := \begin{bmatrix} \mathbf{1}\otimes A & O \\ O & I \end{bmatrix}. \tag{10}$$

*Then, the equilibrium achieving pair of strategies* $[\hat{\eta}^*, \gamma^*]$ *is given by*

$$\hat{\eta}^*(\boldsymbol{t}) = (c^*)'\boldsymbol{t} \text{ and } \gamma^*(\boldsymbol{y}) = b^*\boldsymbol{y}, \tag{11}$$

*where* $c^* \in \mathbb{R}^{(m+1)n}$ *and* $b^* \in \mathbb{R}^n$. *These vectors are given by*

$$c^* = \sqrt{P}\Sigma_t^{-1/2}q \tag{12}$$

*and*

$$b^* = \frac{\rho_{xs}}{\sigma_s^2 + \sigma_w^2}, \tag{13}$$

*where* $\rho_{xs} = \sqrt{P}\Sigma_u\bar{A}'\Sigma_t^{-1/2}q$ *and* $\sigma_s^2 = P$ *are the first and second order statistics of the joint distribution of* $\boldsymbol{x}$ *and* $\boldsymbol{s}$, *and* $q \in \mathbb{R}^{(m+1)n}$ *is the eigenvector of* $W$, *such that* $\|q\| = 1$, *corresponding to the smallest eigenvalue denoted by* $\lambda_{\min}(W)$.

*Furthermore, the outcomes of the game are given by*

$$J_L^* = \lambda_{\min}(W) + G,$$

*where* $G = \begin{bmatrix} \lambda R & O \\ O & (1-\lambda)Q \end{bmatrix}$ *and*

$$J_F^* = \mathrm{Tr}\{\Sigma_x R\} - \frac{P}{P+\sigma_w^2} q'\Sigma_t^{-1/2}\bar{A}\Sigma_u \begin{bmatrix} R & O \\ O & O \end{bmatrix}\Sigma_u\bar{A}'\Sigma_t^{-1/2}q.$$

*Proof.* Let $u := \left[\begin{smallmatrix}x\\ \theta\end{smallmatrix}\right]$ and correspondingly $\boldsymbol{u} := \left[\begin{smallmatrix}\boldsymbol{x}\\ \boldsymbol{\theta}\end{smallmatrix}\right]$ almost everywhere on $\mathbb{R}^{2n}$, and define $\hat{\boldsymbol{u}} := \mathbb{E}\{\boldsymbol{u}|\boldsymbol{y}\}$. Furthermore, let $U := [\,I\ O\,]$ and $D := [\,O\ I\,]$. Then, by (5), we have

$$\min_{\hat{\eta}\in\Omega} \lambda\mathbb{E}\|U\boldsymbol{u} - U\hat{\boldsymbol{u}}\|_R^2 + (1-\lambda)\mathbb{E}\|D\boldsymbol{u} - U\hat{\boldsymbol{u}}\|_Q^2, \tag{14}$$

which can be written as

$$\begin{aligned} \min_{\hat{\eta}\in\Omega} & \ \mathbb{E}\{\boldsymbol{u}'(\lambda U'RU + (1-\lambda)D'QD)\boldsymbol{u}\} \\ & - 2\mathbb{E}\{\boldsymbol{u}'(\lambda U'RU + (1-\lambda)D'QU)\hat{\boldsymbol{u}}\} \\ & + \mathbb{E}\{\hat{\boldsymbol{u}}'(\lambda U'RU + (1-\lambda)U'QU)\hat{\boldsymbol{u}}\}. \end{aligned} \tag{15}$$

[5] Another definition for $V$ is provided at (18). $\mathbf{1} \in \mathbb{R}^{\mathbf{m}}$ is a vector whose terms are 1 and $\otimes$ refers to the Kronecker product.

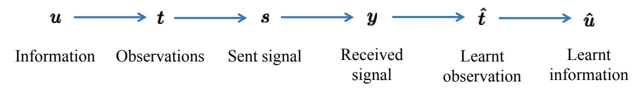


Fig. 2. Evolution of information during transmission from the information provider to the decision maker.

We note that for an arbitrary matrix $\Delta \in \mathbb{R}^{2n\times 2n}$, we have

$$\begin{aligned} \mathbb{E}\{\hat{\boldsymbol{u}}'\Delta\boldsymbol{u}\} &= \mathbb{E}\{\mathbb{E}\{\hat{\boldsymbol{u}}\Delta\boldsymbol{u}|\boldsymbol{y}\}\} \\ &= \mathbb{E}\{\hat{\boldsymbol{u}}'\Delta\mathbb{E}\{\boldsymbol{u}|\boldsymbol{y}\}\} \\ &= \mathbb{E}\{\hat{\boldsymbol{u}}'\Delta\hat{\boldsymbol{u}}\}, \end{aligned} \tag{16}$$

where the first line follows due to the law of iterated expectations, and the second line follows since $\hat{\boldsymbol{u}}$ is a bounded function of $\boldsymbol{y}$ almost everywhere, and given $\boldsymbol{y}$, $\hat{\boldsymbol{u}}$ is deterministic. Note also that the first term in (15) does not depend on the optimization arguments, and hence (5) can be re-written as

$$\min_{\hat{\eta}\in\Omega} \mathbb{E}\{\hat{\boldsymbol{u}}'V\hat{\boldsymbol{u}}\} + G, \tag{17}$$

where

$$V = -\lambda U'RU + (1-\lambda)(U'QU - D'QU - U'QD) \tag{18}$$

and $G := \mathrm{Tr}\{\mathbb{E}\{\boldsymbol{u}\boldsymbol{u}'\}(\lambda U'RU + (1-\lambda)D'QD)\}$ subject to (2).

We first define intermediate parameters $\boldsymbol{t} := \left[\begin{smallmatrix}\boldsymbol{z}\\ \boldsymbol{\theta}\end{smallmatrix}\right]$ and $\hat{\boldsymbol{t}} := \mathbb{E}\{\boldsymbol{t}|\boldsymbol{y}\}$. Then, the following lemma implies that $\boldsymbol{y} \to \hat{\boldsymbol{t}} \to \hat{\boldsymbol{u}}$ form a Markov chain in that order.

**Lemma 1.** *For zero-mean jointly Gaussian parameters* $\boldsymbol{x}$, $\boldsymbol{s}$, *and* $\boldsymbol{y}$ *that form a Markov chain* $\boldsymbol{x} \to \boldsymbol{s} \to \boldsymbol{y}$ *in this order, the conditional expectations with respect to* $\boldsymbol{y}$ *satisfy the following equality:*

$$\mathbb{E}\{\boldsymbol{x}|\boldsymbol{y}\} = \mathbb{E}\{\boldsymbol{x}\boldsymbol{s}\}\mathbb{E}\{\boldsymbol{s}\boldsymbol{s}'\}^{-1}\mathbb{E}\{\boldsymbol{s}|\boldsymbol{y}\}. \tag{19}$$

*Proof.* The proof is provided in Appendix A. ■

By Lemma 1, there exists a matrix $B \in \mathbb{R}^{2n\times(m+1)n}$ such that $\hat{\boldsymbol{u}} = B\hat{\boldsymbol{t}}$, i.e.,

$$B := \mathbb{E}\{\boldsymbol{u}\boldsymbol{t}'\}\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}. \tag{20}$$

Therefore, the random parameters $\boldsymbol{u} \to \boldsymbol{t} \to \boldsymbol{s} \to \boldsymbol{y} \to \hat{\boldsymbol{t}} \to \hat{\boldsymbol{u}}$ form a Markov chain in that order as seen in Fig. 2. The following lemma provides an inequality between the jointly Gaussian random variables that form a Markov chain.

**Lemma 2.** *Let* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}$ *be jointly (possibly multi-variate) Gaussian random parameters with positive-definite covariance matrices* $\Sigma_j$ *for* $j = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}\}$, *joint and cross covariance matrices* $\Sigma_{\iota i}$ *and* $\rho_{ji}$, *respectively, for* $j, i \in \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}\}$ *and* $j \neq i$, *e.g.,*

$$\Sigma_{xt} = \begin{bmatrix} \Sigma_x & \rho_{xt} \\ \rho_{tx} & \Sigma_t \end{bmatrix},$$

*such that they form a Markov chain in the following order:* $\boldsymbol{x} \to \boldsymbol{y} \to \boldsymbol{z} \to \boldsymbol{t}$. *Then, we have the following inequality*[6]*:*

$$\det(I - \rho_{zy}\Sigma_y^{-1}\rho_{yz}\Sigma_z^{-1}) \leq \det(I - \rho_{tx}\Sigma_x^{-1}\rho_{xt}\Sigma_t^{-1}). \quad (21)$$

*Proof.* The proof is provided in Appendix B. ■

We note that since $\hat{\boldsymbol{t}}$ is $\sigma - \boldsymbol{y}$ measurable, $\hat{\boldsymbol{t}}$ is a degenerate Gaussian random vector with a rank-1 covariance matrix $T := \mathbb{E}\{\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}'\}$. To this end, we define $\tilde{\boldsymbol{t}} := \hat{\boldsymbol{t}} + \epsilon\boldsymbol{\mu}$, where $\epsilon > 0$ and $\boldsymbol{\mu} \sim \mathbb{N}(0, I)$, such that $\boldsymbol{t} \to \boldsymbol{s} \to \boldsymbol{y} \to \tilde{\boldsymbol{t}}$ form a Markov chain in that order. Then, we have $I(\boldsymbol{t}; \tilde{\boldsymbol{t}}) \leq I(\boldsymbol{s}; \boldsymbol{y})$ and Lemma 2 yields

$$1 - \frac{\sigma_s^2}{\sigma_s^2 + \sigma_w^2} \leq \det(I - \mathbb{E}\{\tilde{\boldsymbol{t}}\boldsymbol{t}'\}\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}\mathbb{E}\{\boldsymbol{t}\tilde{\boldsymbol{t}}'\}\mathbb{E}\{\tilde{\boldsymbol{t}}\tilde{\boldsymbol{t}}'\}^{-1}),$$

which can also be written as

$$\begin{aligned} 1 - &\frac{\sigma_s^2}{\sigma_s^2 + \sigma_w^2} \\ &\leq \det(I - \mathbb{E}\{\hat{\boldsymbol{t}}\boldsymbol{t}'\}\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}\mathbb{E}\{\boldsymbol{t}\hat{\boldsymbol{t}}'\}(\mathbb{E}\{\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}'\} + \epsilon^2 I)^{-1}). \end{aligned}$$

By the matrix determinant lemma [15], we have

$$\begin{aligned} &\det(I - \mathbb{E}\{\hat{\boldsymbol{t}}\boldsymbol{t}'\}\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}\mathbb{E}\{\boldsymbol{t}\hat{\boldsymbol{t}}'\}(\mathbb{E}\{\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}'\} + \epsilon^2 I)^{-1}) \\ &= \det(I - \mathbb{E}\{\boldsymbol{t}\hat{\boldsymbol{t}}'\}(\mathbb{E}\{\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}'\} + \epsilon^2 I)^{-1}\mathbb{E}\{\hat{\boldsymbol{t}}\boldsymbol{t}'\}\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}). \quad (22) \end{aligned}$$

Note also that $\mathbb{E}\{\boldsymbol{t}\hat{\boldsymbol{t}}'\} = \mathbb{E}\{\mathbb{E}\{\boldsymbol{t}\hat{\boldsymbol{t}}'|\boldsymbol{y}\}\}$ by the law of iterated expectations, which implies that $\mathbb{E}\{\boldsymbol{t}\hat{\boldsymbol{t}}'\} = \mathbb{E}\{\mathbb{E}\{\boldsymbol{t}|\boldsymbol{y}\}\hat{\boldsymbol{t}}'\}$ since $\hat{\boldsymbol{t}}$ is $\sigma - \boldsymbol{y}$ measurable and $\hat{\boldsymbol{t}}$ is a bounded function of $\boldsymbol{y}$ almost everywhere. Therefore, we obtain

$$\mathbb{E}\{\boldsymbol{t}\hat{\boldsymbol{t}}'\} = \mathbb{E}\{\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}'\}. \quad (23)$$

Then, by (22), we have

$$\begin{aligned} \det(I - \mathbb{E}\{\hat{\boldsymbol{t}}\boldsymbol{t}'\}&\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}\mathbb{E}\{\boldsymbol{t}\hat{\boldsymbol{t}}'\}(\mathbb{E}\{\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}'\} + \epsilon^2 I)^{-1}) \\ &= \det(I - T(T + \epsilon^2 I)^{-1}T\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}) \quad (24) \end{aligned}$$

for $\epsilon > 0$. Furthermore, let $T \succeq O$ have a Cholesky decomposition $T = LL'$ such that[7]

$$\begin{aligned} &\lim_{\epsilon \to 0} \det(I - T(T + \epsilon^2 I)^{-1}T'\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}) \\ &\quad = \lim_{\epsilon \to 0} \det(I - LL'(LL' + \epsilon^2 I)^{-1}LL'\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}) \\ &\quad \overset{(a)}{=} \det(I - LL^+LL'\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}) \\ &\quad \overset{(b)}{=} \det(I - T\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}) \\ &\quad \overset{(c)}{=} 1 - \mathrm{Tr}\{T\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}^{-1}\}, \quad (25) \end{aligned}$$

where $(a)$ follows by the limit based definition of pseudo-inverse and since $\det(\cdot)$ is a continuous function, $(b)$ follows since $L = LL^+L$, and $(c)$ follows by the matrix determinant lemma. Therefore, we obtain

$$\mathrm{Tr}\{T\mathbb{E}\{tt'\}^{-1}\} \leq \frac{\sigma_s^2}{\sigma_s^2 + \sigma_w^2}. \quad (26)$$

[6]For the degenerate case, i.e., if the covariance matrix is not full rank, the inequality (21) may not hold since (42) does not hold.

[7]$A^+$ refers to the pseudo-inverse of $A$ [15].

By the power constraint (2), $\sigma_s^2 \leq P$. Since $\sigma_s^2/(\sigma_s^2 + \sigma_w^2)$ is an increasing function of $\sigma_s^2$, we have

$$\frac{\sigma_s^2}{\sigma_s^2 + \sigma_w^2} \leq \frac{P}{P + \sigma_w^2}.$$

Let $\bar{P} := P/(P + \sigma_w^2)$ and $\Sigma_t := \mathbb{E}\{tt'\}$; then, (26) implies that

$$\mathrm{Tr}\{T\Sigma_t^{-1}\} \leq \bar{P}. \quad (27)$$

Furthermore, by $\hat{\boldsymbol{u}} = B\hat{\boldsymbol{t}}$, the main optimization problem (17) can be written as

$$\min_{\hat{\eta} \in \Omega} \mathrm{Tr}\{T\bar{V}\} + G \quad (28)$$

subject to (2), where $\bar{V} := B'VB$. Note that (27) is a *necessary* condition that $T$ should satisfy based on the constraint (2). Then, a lower bound on the original optimization problem is given by

$$\begin{aligned} &\min_{\bar{S} \in \mathbb{S}^{(m+1)n}} \mathrm{Tr}\{\bar{S}\bar{V}\} + G \quad (29) \\ &\text{s.t. } \mathrm{Tr}\{\bar{S}\Sigma_t^{-1}\} \leq \bar{P}, \ \bar{S} \succeq O. \end{aligned}$$

since $T \succeq O$.

Let $S := (1/\bar{P})\Sigma_t^{-1/2}\bar{S}\Sigma_t^{-1/2}$. Then, $\bar{S} = \bar{P}\Sigma_t^{1/2}S\Sigma_t^{1/2}$ and (29) can be written as

$$\begin{aligned} &\min_{S \in \mathbb{S}^{(m+1)n}} \mathrm{Tr}\{SW\} + G \quad (30) \\ &\text{s.t. } \mathrm{Tr}\{S\} \leq 1, \ S \succeq O, \end{aligned}$$

where $W := \bar{P}\Sigma_t^{1/2}\bar{V}\Sigma_t^{1/2}$.

We note that the optimization objective in (30) is linear in the optimization argument while the constraint set $\Phi := \{S \in \mathbb{S}^{(m+1)n} | \mathrm{Tr}\{S\} \leq 1, S \succeq O\}$ is non-empty compact, and convex. Therefore, the global minimum is attained at the *extreme points*[8] of $\Phi$. The following lemma characterizes the extreme points of $\Phi$.

**Lemma 3.** *A point $S_e$ in $\Phi$ is an extreme point if, and only if, $S_e$ has at most one non-zero eigenvalue, which can only be* 1.

*Proof.* The proof is provided in Appendix C. ■

By Lemma 3, the solution $S^*$ of (30) is either a positive semi-definite matrix, which has a single non-zero eigenvalue, whose value is 1; or a zero matrix. This implies that there exists a vector $\zeta \in \mathbb{R}^{(m+1)n}$ such that $\|\zeta\| = 1$ (or 0), and $S^* = \zeta\zeta'$, i.e., $\zeta$ is an eigenvector of $S^*$ corresponding to the eigenvalue 1 (or just a zero vector). Correspondingly, the minimum of (30) yields $\mathrm{Tr}\{\zeta\zeta'W\} = \zeta'W\zeta$ while for all $p \in \mathbb{R}^{(m+1)n}$ such that $\|p\| = 1$, $p'Wp \geq q'Wq$, where $q \in \mathbb{R}^{(m+1)n}$ is the eigenvector of $W$ corresponding to the smallest eigenvalue. This implies that if the smallest eigenvalue of $W$ is non-positive, then $\zeta = q$, and correspondingly, $S^* = qq'$; otherwise, i.e., if the smallest eigenvalue of $W$ is positive, then $S^* = \bar{S}^* = O$.

[8]A point of a convex set is an extreme point, if it cannot be written as a convex combination of any other points in the set.

We note that the case when the smallest eigenvalue of $W$ is positive implies that $W$ is a positive-definite matrix. However, since $W = \bar{P}\Sigma_t^{1/2}B'VB\Sigma_t^{1/2}$ and $\Sigma_t$ is non-singular, $W$ and $B'VB$ are congruent matrices and $B'VB$ has a rank at most $2n$ due to $V \in \mathbb{S}^{2n}$, i.e., $B'VB$ is singular, and therefore, not a positive-definite matrix. Sylvester's law of inertia [16] implies that $W$ and $B'VB$ have the same positive and negative indices of inertia, i.e., they have the same number of positive, negative, and zero eigenvalues. Correspondingly, $W$ is not a positive-definite matrix, and therefore, the smallest eigenvalue of $W$ is not positive.

Since $\hat{\eta} \in \Omega$, there exists a vector $c \in \mathbb{R}^{(m+1)n}$ such that $s = \hat{\eta}(t) = c't$. Then, the decision maker receives $y = c't + w$ and correspondingly, we have

$$\hat{\boldsymbol{t}} = \frac{\mathbb{E}\{\boldsymbol{t}\boldsymbol{y}\}}{\mathbb{E}\{\boldsymbol{s}^2\} + \sigma_w^2}\boldsymbol{y} = \frac{\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}c(c'\boldsymbol{t} + \boldsymbol{w})}{c'\mathbb{E}\{\boldsymbol{t}\boldsymbol{t}'\}c + \sigma_w^2}. \tag{31}$$

This leads to

$$\mathbb{E}\{\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}'\} = \frac{\Sigma_t c c' \Sigma_t}{c'\Sigma_t c + \sigma_w^2}. \tag{32}$$

Note that if we select $c = \sqrt{P}\Sigma_t^{-1/2}q$, which also satisfies the power constraint (2), then by (31), we obtain $T = \bar{P}\Sigma_t^{1/2}qq'\Sigma_t^{1/2}$, and substituting it in (28) yields

$$\mathrm{Tr}\{\bar{P}\Sigma_t^{1/2}qq'\Sigma_t^{1/2}\bar{V}\} = \mathrm{Tr}\{qq'W\} = \lambda_{\min}(W), \tag{33}$$

which leads to the global minimum of the lower bound, i.e., $\lambda_{\min}(W) \leq 0$. Therefore, we conclude that the best deception strategy is given by $\boldsymbol{s} = (c^*)'\boldsymbol{t}$, where $c^* = \sqrt{P}\Sigma_t^{-1/2}q$. Correspondingly, by (20) and (31), the best decision is given by $\hat{\boldsymbol{u}} = b^*\boldsymbol{y}$, where

$$b^* = \mathbb{E}\{\boldsymbol{u}\boldsymbol{t}'\}\Sigma_t^{-1}\frac{\sqrt{P}\Sigma_t\Sigma_t^{-1/2}q}{P + \sigma_w^2}.$$

Note that $\boldsymbol{z}$ can be written as

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_1 \\ \vdots \\ \boldsymbol{z}_m \end{bmatrix} = (\mathbf{1} \otimes A)\boldsymbol{u} + \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_m \end{bmatrix}, \tag{34}$$

where $\mathbf{1} \in \mathbb{R}^m$ denotes the vector whose entries are all 1, $\otimes$ refers to the Kronecker product [16], and $\boldsymbol{v}_i$'s have independent and identical distribution, which is the same with $\boldsymbol{v}$. Then, the cross correlation between $\boldsymbol{u}$ and $\boldsymbol{t}$ is given by

$$\mathbb{E}\{\boldsymbol{u}\boldsymbol{t}'\} = \mathbb{E}\{\boldsymbol{u}\boldsymbol{u}'\}\begin{bmatrix} \mathbf{1}\otimes A & O \\ O & I \end{bmatrix}'. \tag{35}$$

Furthermore, the outcome of the game for the information provider is given by

$$J_L^* = \lambda_{\min}(W) + G \tag{36}$$

while the outcome of the game for the decision maker is given by

$$\begin{aligned} J_F^* &= \mathbb{E}\|U\boldsymbol{u} - U\hat{\boldsymbol{u}}^*\|_R^2, \\ &= \mathbb{E}\{\boldsymbol{u}'U'RU\boldsymbol{u}\} - 2\mathbb{E}\{\boldsymbol{u}'U'RU\hat{\boldsymbol{u}}^*\} + \mathbb{E}\{(\hat{\boldsymbol{u}}^*)'U'RU\hat{\boldsymbol{u}}^*\}, \\ &\overset{(a)}{=} \mathrm{Tr}\{\Sigma_x R\} - \mathrm{Tr}\{\mathbb{E}\{\hat{\boldsymbol{u}}^*(\hat{\boldsymbol{u}}^*)'\}U'RU\}, \\ &= \mathrm{Tr}\{\Sigma_u U'RU\} - \mathrm{Tr}\{T^*B'U'RUB\}, \\ &\overset{(b)}{=} \mathrm{Tr}\{\Sigma_u U'RU\} - \bar{P}q'\Sigma_t^{1/2}B'U'RUB\Sigma_t^{1/2}, \end{aligned}$$

where $(a)$ follows due to the law of iterated expectations and $(b)$ follows by (20) and (35). This completes the proof. □

**Remark 1.** *We point out that in Reference [12], the author addresses the multi-dimensional information disclosure in strategic environments without a channel and shows the optimality of linear strategies within the general class of strategies. However, here, we consider an additive noise channel between the information provider and the decision maker. When there is a channel in between, even in the non-strategic environments, where the information provider and the decision maker have the same objective, linear strategies are not optimal within the general class of strategies in multi-variate Gaussian information disclosure in general [17].*

**Remark 2.** *When $\lambda = 1$ and $R = Q = I$, both agents have the same objective, which is to minimize $\mathbb{E}\{\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|^2\}$. Then, if the information provider has perfect access to the information $x$, i.e., $z = x$, Theorem 1 also implies that the optimal linear strategy is $s = \sqrt{P/\lambda_{\max}(\Sigma_x)}q_x'x$, where $q_x$ is the eigenvector of $\Sigma_x$, such that $\|q_x\| = 1$, corresponding to $\lambda_{\max}$, as also shown in [18].*

**Remark 3.** *If the information provider has perfect access to the information $x$, i.e., $z = x$ and $t = u$, and there is a perfect channel, where the information provider can transmit a vector of dimension $\mathbb{R}^{2n}$, then Reference [12] shows that the optimal information disclosure strategy is given by $s_o = (C_o^*)'t$, where $C_o \in \mathbb{R}^{2n\times r}$ is given by*

$$C_o^* = \Sigma_t^{-1/2}Q_-, \tag{37}$$

*and $Q_- = \begin{bmatrix} q_1 & \cdots & q_r \end{bmatrix}$ is the eigenvectors of $W_o = \Sigma_t^{1/2}V\Sigma_t^{1/2}$ associated with the negative eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r$. Correspondingly, when the information provider has perfect access to the information $x$, we show that if there is a scalar channel, then by Theorem 1, the optimal information disclosure strategy would be given by $s = (c^*)'t$, where*

$$c^* = \sqrt{P}\Sigma_t^{-1/2}q, \tag{38}$$

*and $q \in \mathbb{R}^{2n}$ is the eigenvector of $W = (P/(P + \sigma_w^2))\Sigma_t^{1/2}V\Sigma_t^{1/2}$ associated with the minimum negative eigenvalue, given by $(P/(P + \sigma_w^2))\lambda_1$. Note that $q$ is also the eigenvector of $\Sigma_t^{1/2}V\Sigma_t^{1/2}$ associated with the minimum negative eigenvalue since $P/(P+\sigma_w^2) > 0$. Therefore, under the linearity constraint on the strategies of the information provider, there does not exist game-channel separation such that we can compute the equilibrium achieving strategy pair $\hat{\eta}^*, \gamma^*$ as if there is a perfect channel between the information provider and the decision maker, and then transmit $\hat{\eta}^*(\boldsymbol{t})$ to mitigate the impact of the channel. Therefore the game and the channel should be considered jointly. However, interestingly, $c^*$ is the first column of $C_o^*$ that is scaled by $\sqrt{P}$ to satisfy the power constraint* (2).

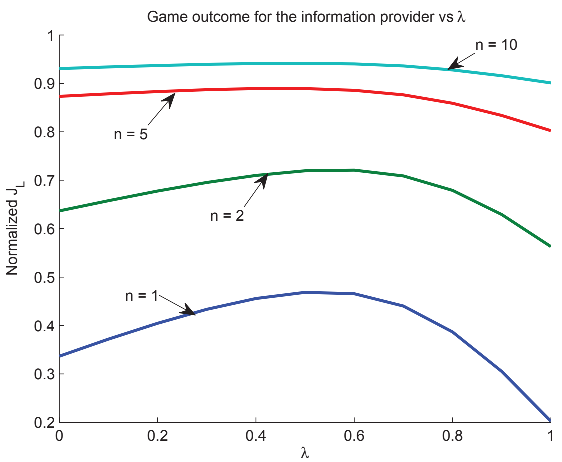


Fig. 3. Normalized game outcome of the information provider (the leader) versus $\lambda$ compared over various dimensional information. Note that $\lambda = 0$ implies the most deceptive, while $\lambda = 1$ implies cooperative, information provider.

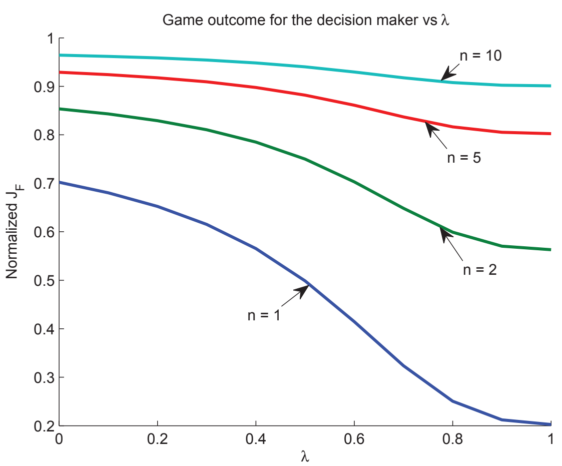


Fig. 4. Normalized game outcome of the decision maker (the follower) versus $\lambda$ compared over various dimensional information. Note that the decision maker's outcome matches with the information provider's outcome when $\lambda = 1$, which is the case where both players have the same objective.

## IV. ILLUSTRATIVE EXAMPLES

As numerical illustrations, we examine the impact of $\lambda$, e.g., $\lambda = 0, 0.1, \ldots, 1$, on the outcomes of the game for both information provider and decision maker when $n = 1, 2, 5$, and 10. We set $m = 1$, $\Sigma_v = 0.1\, I$, $A = I$, $P = 1$, and $\sigma_w^2 = 0.2$. We construct the joint covariance matrices of $\boldsymbol{x}$ and $\boldsymbol{\theta}$ randomly as follows. We draw a number from the uniform distribution from 0 to 1 for each entry of a matrix $D$. Then, we can construct a positive-definite covariance matrix by $\Sigma_t = (D + D')/2 + 2n\, I$, where the last term ensures that the constructed matrix is diagonally dominant, and therefore, positive-definite. For example, for $n = 1$, we have obtained $\begin{bmatrix} 2.3233 & 0.5369 \\ 0.5369 & 2.0456 \end{bmatrix}$.

While examining the game outcomes for various dimensional information, we normalize the objective functions by $\mathrm{Tr}\{\Sigma_x\}$ for illustrative purposes. Furthermore, if the information provider does not provide any information to the decision maker, the decision maker could achieve $\mathrm{Tr}\{\Sigma_x R\}$ by estimating the underlying information as a zero vector. Therefore, with any additional related information, the decision maker should outperform that. Note that we have set $R = I$, therefore, the normalized game outcome would be 1 if the information provider did not disclose any information.

In Figs. 3 and 4, we plot the outcomes of the game versus $\lambda \in [0, 1]$ and compare for different size information, e.g., $n = 1, 2, 5$, and 10. In the numerical examples, we have observed that the outcome is worse when the information provider is the most deceptive than the case he/she is cooperative. This can be attributed to the difficulty of manipulating the decision maker's perception about the underlying information while being truthful about the disclosed information. Additionally, the outcome of the information provider peaks at certain $\lambda$, which can be observed more clearly at small dimensional information disclosure, e.g., $n = 1, 2$. On the other side, the outcome for the decision maker is always worse when the information provider is deceptive. However, since the information is transmitted over a scalar channel, the impact of deception relatively decreases as the dimension of the information increases. This implies that when the quality of information transmission decreases the ability to deceive the decision maker also decreases. Correspondingly, intuitively, in order to avoid effective manipulation on the decisions made, the decision maker could prefer to acquire information from different providers over channels with low signal-to-noise ratio and process centrally instead of acquiring the information only from one provider at high quality. We also note that when $\lambda = 1$, which is the classical communication case where the players are cooperating, the outcomes of both players match.

## V. CONCLUSION

In this paper, we have addressed the deceptive multi-dimensional information disclosure problem over a Gaussian channel between a deceptive information provider and a decision maker, which can be viewed as a Stackelberg game, where the provider is the leader, due to the differences between the objectives of the provider and the decision maker, and the reputation constraint for the provider to sustain his/her business. For multi-variate Gaussian information, additive Gaussian noise channel, noisy observations, and quadratic (different) loss functions, we have analytically formulated optimal (linear) deception strategies and optimal decision strategies, and the corresponding outcomes of the game. We have numerically examined the impact of the deception and the dimension of the underlying information on the outcomes of the players.

Some future directions of research on this topic include analysis of deception strategies in a competitive environment where there are multiple information providers and the corresponding optimal information acquisition strategies for the decision maker for unmanipulated decisions, and also the formulation of the optimal deception and decision strategies over vector channels.

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### A. *Proof of Lemma 1*

Since the parameters are zero-mean jointly Gaussian the conditional expectations with respect to the random variable $\boldsymbol{y}$ are given by

$$\mathbb{E}\{\boldsymbol{x}|\boldsymbol{y}\} = \mathbb{E}\{\boldsymbol{xy}\}\mathbb{E}\left\{\boldsymbol{yy}'\right\}^{-1}\boldsymbol{y} \tag{39}$$

$$\mathbb{E}\{\boldsymbol{s}|\boldsymbol{y}\} = \mathbb{E}\{\boldsymbol{sy}\}\mathbb{E}\left\{\boldsymbol{yy}'\right\}^{-1}\boldsymbol{y}. \tag{40}$$

Yet, the correlation between $\boldsymbol{x}$ and $\boldsymbol{y}$ can be calculated as

$$\begin{aligned}\mathbb{E}\{\boldsymbol{xy}\} &= \int\int xyf_{\boldsymbol{xy}}(x,y)dxdy\\ &= \int\int\int xyf_{\boldsymbol{x}}(x|s)f_{\boldsymbol{sy}}(s,y)dsdxdy,\end{aligned}$$

where $f$ denotes the (joint) probability density function of the designated random variable(s), due to the formed Markov chain. Then,

$$\begin{aligned}\mathbb{E}\{\boldsymbol{xy}\} &= \int\int \mathbb{E}\{\boldsymbol{x}|\boldsymbol{s}=s\}yf_{\boldsymbol{sy}}(s,y)dsdy\\ &= \mathbb{E}\{\boldsymbol{xs}\}E\{\boldsymbol{ss}'\}^{-1}\int\int syf_{\boldsymbol{sy}}(s,y)dsdy. \end{aligned}\tag{41}$$

By (39) and (40), (41) yields (19). This completes the proof of the lemma.

### B. *Proof of Lemma 2*

By data processing inequality [19], we have $I(\boldsymbol{x};\boldsymbol{t}) \leq I(\boldsymbol{y};\boldsymbol{z})$. Since all the parameters are jointly Gaussian, the mutual information has a closed form expression in terms of second order statistics and we obtain

$$\frac{1}{2}\log\left(\frac{\det(\Sigma_x)\det(\Sigma_t)}{\det(\Sigma_{xt})}\right) \leq \frac{1}{2}\log\left(\frac{\det(\Sigma_y)\det(\Sigma_z)}{\det(\Sigma_{yz})}\right).$$

Through the use of the Schur complement to compute determinant [20], we have

$$\det(\Sigma_{xt}) = \det(\Sigma_x)\det(\Sigma_t - \rho_{tx}\Sigma_x^{-1}\rho_{xt}) \tag{42a}$$

$$\det(\Sigma_{yz}) = \det(\Sigma_y)\det(\Sigma_z - \rho_{zy}\Sigma_y^{-1}\rho_{yz}), \tag{42b}$$

which lead to (21).

### C. *Proof of Lemma 3*

Since every matrix in $\Phi$ is a positive semi-definite matrix, the zero matrix $O$ is an extreme point of $\Phi$. Furthermore, let $P$ be a matrix in $\Phi$ such that $P$ has only one non-zero eigenvalue, whose value is one. Suppose that there exist two other matrices $M,N \in \Phi$ such that $P = tM + (1-t)N$ for some $t \in (0,1)$. Let $p_1, p_0 \in \mathbb{R}^{(m+1)n}$ be eigenvectors of $P$ corresponding to the eigenvalue 1 and an eigenvalue 0. Note that the eigenvalues of every matrix in $\Phi$ are bounded by 0 and 1 since $\mathrm{Tr}\{M\} \leq 1$ while $M \succeq O$ for all $M \in \Phi$. However, the convex combination implies also that $tp_1'Mp_1 + (1-t)p_1'Np_1 = p_1'Pp_1 = 1$ and $tp_0'Mp_0 + (1-t)p_0'Np_0 = p_0'Pp_0 = 0$. Therefore, $p_1'Mp_1 = p_1'Np_1 = 1$ and $p_0'Mp_0 = p_0'Np_0 = 0$, which yields that $p_1$ and $p_0$ are the eigenvectors of $M$ and $N$. Since $p_0$ could be the eigenvector of any arbitrarily chosen eigenvalue 0, the matrices $P, M,$ and $N$ have the same eigenvectors and eigenvalues, i.e., $P = M = N$, which leads to a contradiction.

Conversely, any other matrix $R \neq O$, which does not have a single non-zero eigenvalue 1, in $\Phi$, is not a extreme point. Consider matrices $M$ and $N$ that have the same eigenvectors with $R$. Since $R \neq O$, $R$ has at least one positive eigenvalue that is less than 1, say $\lambda \in (0,1)$. Let $M$ and $N$ also have the same corresponding eigenvalues with $P$ except $\lambda$. Furthermore, let $M$ have $\lambda+\epsilon$ and $N$ have $\lambda-\epsilon$ as eigenvalues such that $\lambda\pm\epsilon \in [0,1]$. Then, we have the convex combination $R = 1/2M + 1/2N$ while $M \neq R$ and $N \neq R$, which implies that $R$ is not an extreme point of $\Phi$.